APPROXIMATE SOLUTION TECHNIQUES
FOR RANDOMLY EXCITED NON-LINEAR
SYSTEMS

A Dissertation Submitted in Partial Fulfilment of the Requirements
for the Master Degree in

Earthquake Engineering & Engineering Seismology

By

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Name of Reviewer 1

Name of Reviewer 2
ABSTRACT

Randomly excited non-linear dynamic systems are frequently met in engineering practice. The source of randomness may arise from earthquake ground motion or wind or wave motions at sea exciting offshore structures. The objectives are investigation and thorough understanding of stochastic non-linear phenomena as well computation of certain non-linear response characteristics.

The excitations, that will be studied, are stationary Gaussian processes. These processes can be white noise processes or processes with band limited frequency spectra. Both models for stationary and non-stationary have been discussed with a short description of power spectral density (PSD) function for random processes.

The primary motive of this study to present thorough investigation of approximate techniques for estimating the stationary probability density function (PDF) of the response of nonlinear system subjected to additive Gaussian white noise excitations. Attention is focused on the exponential closure method in which the probability density function can be approximated with the exponential function of polynomial in state variables. Special measure is taken to satisfy Fokker-Planck-Kolmogorov (FPK) equation in the weak sense of integration with the assumed PDF. Several nonlinear oscillators are under additive white noises were solved. Method is also extended for higher order of assumed PDF. The response obtained from numerical results showed close approximation with exact PDF regardless of degree of system non-linearity. Higher order assumed PDF coincide with exact ones regardless of degree of non-linearity in the system,

Keywords: Gaussian white noise; Approximation techniques; FPK equation; closure methods
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\( A(\omega) \) = the Fourier amplitude spectrum of excitation
\( A_j \) = first derivative moment coefficient in FPK equation
\( B_{jk} \) = second order derivative moment coefficient in FPK equation
\( G_j \) = the probability flow in \( j^{th} \) direction
\( F_X(x) \) = the probability distribution function of a random variable \( X \)
\( f_X(x) \) = the probability density function of random variable \( X \)
\( H(t) \) = heaviside unit step function
\( E[.\] \) = expected value operator
\( k_e \) = equivalent stiffness of replacing system
\( S_{aa}(\omega) \) = the spectral density function at frequency \( \omega \)
\( R_{XX}(t) \) = autocorrelation function of stochastic process \( X(t) \)
\( I_0 \) = total energy of random excitation
\( W(t) \) = Gaussian white noise

Greek symbols
\( \beta_e \) = equivalent damping in the replacing system
\( \phi(\lambda) \) = probability potential of replacement system at amplitude of \( \lambda \)
\( \mu_{Xp} \) = \( p^{th} \) statistical moment of random variable \( X \)
\( \varepsilon \) = degree of non-linearity in the system

Abbreviations
CNC = cumulant neglect closure
FPK = Fokker Planck Kolmogorov
MDOF = multiple degree of freedom
PDF = probability density function
psd = power spectral density
1. INTRODUCTION

1.1 Objectives

The class of systems, which defines the scope of interest, can be characterized by the following features:

- Polynomial, non-polynomial, and even discontinuous (non-smooth) nonlinearities;
- Multi-Degree-Of-Freedom (MDOF)-systems;
- Strong nonlinearities;
- Stationary systems.

The choice to aim for methods applicable to systems with a general form of nonlinearity stems from practical considerations. In practice, namely, the form of the nonlinearity is rarely of a purely polynomial kind. Furthermore, discontinuous nonlinearities, which will be studied extensively throughout this thesis, can be found in many engineering systems, for example ships colliding against fenders on quay sides, snubbers in solar panels on satellites, suspension bridges, stops in axle suspensions for vehicles and so on. Moreover, most engineering systems essentially are MDOF systems. However, it should be noted that, when complex nonlinearities are considered, it can be advisable to study a Single-Degree-Of-Freedom (SDOF)-system first. This is surely the case when one is attempting to understand specific, nonlinear, stochastic response phenomena. The decision not to limit ourselves to merely moderate nonlinearities is initiated by the desire to observe, investigate, and understand truly nonlinear behaviour. The class of stochastic excitations can be defined by the following characteristics:

1. Gaussian excitations;
2. Excitations with broad-banded as well as narrow-banded spectra, either white or non-white;
3. External as well as parametric excitations;
4. Stationary excitations.

In most practical situations, the random inputs can be accurately modelled as normally distributed processes; so, the deviation from normality is not significant. Hereby, we have, thus, imposed a restriction on the probability density function of the excitations. However, a wide variety of spectral energy distributions for the excitations is incorporated within the research scope. Most excitations, which stem from environmental loads acting on mechanical systems, can be modelled by broad banded random processes. Situations with parametric excitations can easily be encountered when one of the system parameters varies randomly in time.

The desired ‘steady state characteristics’ are

1. Statistical moments and the probability density function;
2. Power spectral density.

In literature, the investigation of the stochastic response of dynamic systems is often merely focused on the statistical moments and the probability density function. Consequently, essential information, regarding the dynamic response, remains hidden. Therefore, it is of the utmost importance that the response information in

The frequency domain, the power spectral density function, is incorporated in the investigations. It will become clear, throughout this thesis, that the information in the frequency domain exhibits many interesting, nonlinear, stochastic phenomena. These frequency domain response characteristics can, moreover, help to understand the root of specific tendencies in the statistical moments of the response.

As mentioned earlier, the research scope is restricted to both stationary (time independent) systems and stationary excitations. We, therefore, are merely interested in the steady state behaviour of the systems. In the following subsections, several existing methods will be discussed, with respect to their suitability to tackle the problems described above. In the majority of the methods, that will be described, the assumption is made that the excitation can be idealized as Gaussian white noise. Realizations of a white noise process at two distinct points in time are independent by definition, no matter how small the time interval between them. A more formal definition of a white noise process is given in chapter 2. When the excitation is a white noise process, the response of the system can be represented by a Markov
process. As a consequence, the unnormalized probability density function of the response is governed by a partial differential equation, called the Fokker-Planck-Kolmogorov (FPK) equation [Caughey, 1963a; Lin, 1967; Melsa and Sage, 1973]. The vast majority of the existing methods make use of the Markov process assumption. In practical applications, the justification of the Markov process assumption is usually based on the following sufficient condition: the increments of the response, during two non overlapping time intervals, are independent events. This ideal property can never be found in a real physical process. However, when the time increments are viewed as observation time laps, the length of these time laps can be chosen to ensure the independence of the increments. On the other hand, one will have to choose the observation time intervals small enough in order to avoid loss of essential information on the dynamics of the system. As long as there is randomness in a real physical process, it is possible to select a long enough observation time interval to ensure Markov-like appearance of the observed increments.
2 STOCHASTIC MODELLING OF STRONG EARTHQUAKE GROUND MOTION

A successful antiseismic study should be based on knowledge of all potential earthquake excitation that could strike the site of a structure in consideration. Theoretically, the class of possible ground motion could be determined if information were available regarding the local conditions, material properties, neighbouring fault system and the nature of expected fault rupture processes. The lack such information and the complexity of the analytical problem that must be solved make this approach practically impossible to achieve.

Alternatively empirical methods can be applied in order to define appropriate models for seismic excitations. It appears that the deterministic models cannot be used for this purpose unless an analytical approach is applied to calculate the displacement field from a given fault mechanisms. Defining the excitation as belonging to a general class of signals with prescribed time and frequency domain properties (duration, peak acceleration, total energy, energy distribution over the frequency range etc), appears as more suitable approach. Stochastic models of the seismic excitation are an example of such an approach, and often have been used in examining the seismic response of structural systems.

The only limitation of stochastic models is that the analytical methods for the analysis of nonlinear system subjected to random excitation are far more complex than the case of a deterministic input. For this reason, the mathematical description of stochastic models should be relatively simple.

Housner [1947] was the first to suggest that “acceleration recorded of earthquakes exhibits characteristics of randomness”. Three types of basic models and their minor variations reflecting increasing level of complexity have been used to model earthquake ground acceleration:
Chapter 2. Stochastic Modelling of Strong Earthquake Ground Motion

(i) White Noise [Housner, 1947; Byrcroft, 1960; Hudson, Housner and Caughey, 1960]
(ii) Stationary Process [Tajimi, 1960; Housner and Jennings, 1964] and
(iii) Nonstationary process [Bolotin, 1960; Cornell, 1964; Shinozuka and Sato, 1967; Amin and Ang, 1968; Boore, 1983]

2.1 Stationary Models

Stationary model have often been used for representation of the frequency content of long duration seismic ground motion. Consider a input earthquake acceleration, \( \ddot{a}(t) \), of duration \( \tau_0 \). The Fourier amplitude spectrum of the excitation, \( A(\omega) \) and the total energy, \( I_0 \), are defined as

\[
A(\omega) = \left| \int_{-\infty}^{\infty} a(t) \exp^{-i\omega t} \, dt \right| = \left| \int_{0}^{\tau_0} a(t) \exp^{-i\omega t} \, dt \right|  
\]

and

\[
I_0 = \int_{0}^{\tau_0} a^2(t) \, dt  
\]

It can be proven by application of the Parsevals’s relation that

\[
I_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} A^2(\omega) \, d\omega  
\]

From equation (2.3) it can be noticed that the Fourier amplitude spectra, \( A(\omega) \), describes the distribution of excitation energy over the frequency domain.

For a stationary stochastic model, the frequency content of the excitation is given by the power spectral density, \( S_{aa}(\omega) \). The relationship between \( S_{aa}(\omega) \) and \( A(\omega) \) can be established in the case of a long duration strong ground motion. acceleration as follows. For a weakly stationary random process, \( a(t) \), the spectral density function, \( S_{aa}(\omega) \), is defined as

\[
S_{aa}(\omega) = \lim_{t_0 \to \infty} \frac{1}{2\pi t_0} E \left[ \left| A^{t_0}(\omega, t_0) \right|^2 \right]  
\]

where \( A^{t_0}(\omega, t_0) \) is the finite Fourier transform over a record of length \( t_0 \), or

\[
A^{t_0}(\omega, t_0) = \int_{0}^{t_0} a(t) \exp^{-i\omega t} \, dt  
\]

Equations (2.1) and (2.4) imply that

\[
S_{aa}(\omega) = \lim_{t_0 \to \infty} \frac{1}{2\pi t_0} E[A^2(\omega)]  
\]

which suggest that spectral density function \( S_{aa}(\omega) \) is proportional to the mean square value of the excitation Fourier amplitude spectrum.

A stationary stochastic model of seismic excitation is defined by specifying the spectral density function, \( S_{aa}(\omega) \). As can be seen from equation (2.4), \( S_{aa}(\omega) \) is given by the
analytical amplitude spectrum, $A(\omega)$, of actual earthquake ground motion. Boore [1983], by analyzing the available records of seismic ground motion acceleration, obtained the following expression for $A(\omega)$

$$A(\omega) = \frac{CM_0}{R} \frac{\omega^2}{1+(\omega/\omega_0)^2} \left[ 1 + (\omega/\omega_m)^2 \right]^{-1/2} e^{-\omega R/2Q\beta} \quad (2.5a)$$

where $M_0$, $\omega_0$, $\omega_m$, $s$, $Q$, $\beta$, and $R$ are model parameters. From equation (2.5a), the asymptotic behaviour for $A(\omega)$ is obtained as

$$A(\omega) \sim \omega^2 \quad \text{as} \quad \omega \rightarrow 0 \quad (2.5b)$$

and

$$A(\omega) \sim e^{-\gamma \omega} \quad \text{as} \quad \omega \rightarrow \infty \quad (2.5c)$$

For application to the nonlinear random response of structural systems, the stochastic excitation model should be mathematically tractable. Analytical results in the theory of random vibrations can be obtained primarily for the case of Gaussian white noise excitation. For this reason model discussed in this section are obtained by appropriate filtering of a white noise excitation. An example of this kind of model is Kanai-Tajimi model. Following analytical expression due to Tajimi [1960] based on work of Kanai [1957], has extensively used to represent power spectra ground l density of acceleration.

$$\Phi(\omega) = \Phi_0 \frac{1+4\zeta_g^2(\omega/\omega_g)^2}{[1-(\omega/\omega_g)]^2 + 4\zeta_g^2(\omega/\omega_g)^2} \quad (2.6)$$

A filtered white noise with power spectral density given by equation (1.17) can be generated by second order filter.

$$\ddot{U} + 2\omega_g \zeta_g \dot{U} + \omega_g^2 U = W(t) \quad (2.7)$$

Where $W(t)$ is white noise with power spectral density $S_0$. From analysis of actual record, it is found that $\zeta_g = 0.6$ and $\omega_g = 5$ correspond closely to the spectral properties for firm ground.

For a specific site parameters $\zeta_g$ and $\omega_g$ should be chosen suitably to represent local site conditions.

The variance and other spectral parameters for Kanai-Tajimi spectrum are given by

$$\sigma_g^2 = \frac{\pi \Phi_0 \omega_g (1+4\zeta_g^3)}{2\zeta_g} \quad (2.8a)$$

$$\Omega_g = \left( \frac{\lambda_g^2}{\lambda_0} \right)^{1/2} = 2.1\omega_g \quad (2.8b)$$

$$\delta_g = \left( 1 + \frac{\lambda_1^2}{\lambda_0^2} \right)^{1/2} = 0.67 \quad (2.8c)$$
In equation (2.8a), (2.8b) and (2.8c), the cut-off frequency $\omega_0 = 4\omega_g$ and $\delta_g = 0.6$. [Vanmarcke 1977]

### 2.2 Nonstationary Models

Stationary models can only be used for the case of long duration excitation, for which the transient information is of no significance. In order to account for the transient nature of the seismic excitation, the time modulation of a stationary process may be considered. To understand Nonstationary models, consider $Z_i(t)$, $i = 1, 2, 3$, represent the three component of ground displacement at a point due to earthquake. Consistent with general characteristics of strong motion earthquakes, each component of the ground acceleration can be expressed as-

$$Z_i = A_i(t)Y_i(t) ; \quad 0 \leq t \leq T$$

$$= 0 \quad \text{otherwise}$$

where $i = 1, 2$ and $3$; $A_i(t)$ are slowly varying random functions of time called envelope; $Y_i(t)$ are a segment of random stationary processes; and $T$ is the duration of the motion.

The nonstationarity in the ground acceleration records arises primarily through envelope functions $A_i(t)$ in equation (2.9). These are the slowly varying random function of time. A simple and adequate nonstationary model can be constructed by assuming $A_i(t) = A(t)$ to be deterministic function and $Y_i(t)$ a set stationary random process with specified power spectral densities. The model may be constructed either by multiplying a segment of filtered white noise by a modulating function (MFWM), or alternatively, by multiplying white noise by a modulating function first and then filtering the product process (FMWM). If the filter characteristics and the modulating functions are identical for two models, the difference in their characteristics depends on the smoothness of $A(t)$. For earthquake long quasi-stationary motion both models yield similar characteristics.

Several functions have been used to model the envelope of ground acceleration records. Following expressions are commonly used:

$$A(t) = (A_0 + A_1 t) e^{-at^n} H(t) ; \quad n = 1 \text{ or } 2, \quad a > 0 \quad \text{[Iyenger and Iyenger, 1969]} \quad (3.0a)$$

$$A(t) = A (e^{-at} - e^{-bt}) H(t), \quad b > a > 0 \quad \text{[Shinozuka and Sato, 1967]} \quad (3.0b)$$

$$A(t) = A \left( \frac{1}{t_1} \right)^a H(t), \quad 0 \leq t \leq t_1$$

$$= A H(t - t_1), \quad t_1 \leq t \leq t_2 \quad \text{Jennings et al., 1969} \quad (3.0c)$$

$$= A \exp [-b (t - t_2)] H(t - t_2) \quad t_2 \leq t \leq t_3$$
\[
A(t) = A_0 \left[ 0.05 \mathcal{C} + d (T - t)^2 \right] H(t - t_3) \quad t \geq t_3
\]

where \( H(t) \) is a heaviside unit step function.

The envelope function represented by (2.9c) covers significant features of a complete range of recorded earthquake ground accelerations. Jeaning et al.[1969] have suggested suitable values for the parameters of the envelope function to model four different types of earthquake motions (A,B,C and D) which are considered to be significant for engineering structures. The values are given in table (2.1). The non-stationary nature of ground motion has a significant influence on tail value of probability estimates, nonlinear response and soil behaviour. Further, the stationary models are inadequate for small near-field earthquake ground motions.

**Table 2.1 Parameters of Envelope function \( A(t) \) in (3.0c)**

<table>
<thead>
<tr>
<th>Type</th>
<th>Target Magnitude</th>
<th>Target Spectrum Intensity</th>
<th>Duration (sec.)</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \geq 8 )</td>
<td>1.5 (El-Centro,1940)</td>
<td>120</td>
<td>4</td>
<td>35</td>
<td>80</td>
<td>2</td>
<td>0.0357</td>
<td>1</td>
<td>0.938x10^{-4}</td>
</tr>
<tr>
<td>B</td>
<td>6 - 8</td>
<td>El-Centro, 1940</td>
<td>50</td>
<td>4</td>
<td>15</td>
<td>30</td>
<td>2</td>
<td>0.0992</td>
<td>1</td>
<td>0.5x10^{-2}</td>
</tr>
<tr>
<td>C</td>
<td>5.5 - 6</td>
<td>Golden Gate,</td>
<td>12</td>
<td>2</td>
<td>4</td>
<td>-</td>
<td>2</td>
<td>0.268</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>D</td>
<td>4.5 – 5.5</td>
<td>Parkfield 1966</td>
<td>10</td>
<td>2</td>
<td>2.5</td>
<td>3.5</td>
<td>2</td>
<td>1.606</td>
<td>2</td>
<td>0.237x10^{-2}</td>
</tr>
</tbody>
</table>
3 RANDOM PROCESSES

The theory of random processes has evolved as a generalization of the concept of random variable. In many problems the outcome of a event is not a real number but a function of one or more parameters, such as time or space or both. In such cases the outcome of each trial is called a realization or a sample function and the collection of all possible functions is called the ensemble of the random process. The random processes have found increasing application as models of a large class of natural phenomena. Some examples of phenomena modelled as random processes are

1. Particles in suspension undergoing Brownian motion as function of time,
2. Ground motion a point during earthquakes as function of time,
3. Unevenness of road surface as a function of distance along the centre line, and
4. The pressure field due to jet noise as function of both the time and space coordinates

In the literature, random processes are also called random function, stochastic processes, and time series if the indexing parameter is time. In proceeding section, we are going to introduce the concept of stochastic processes.

3.1 Stochastic Processes

Before we go any further, the concept of a stochastic process should be introduced. It is important to clarify the difference between as random variable and a stochastic (random) process. A random variable $X$ (on $\mathbb{R}$) is defined by its probability distribution; the probability distribution of $F_X(x)$ of a random variable $X$ can be defined as

$$F_X(x) = \text{probability } (X \leq x) = P(X \leq x) \quad (3.1)$$

Moreover, the probability density function $f_X(x)$ of a random variable $X$ can be defined as

$$f_X(x) = \frac{dF_X(x)}{dx} \quad (3.2)$$
Now the $p^{th}$ statistical moment $\mu_{X_p}$ ($p=1,2,3,\ldots$) of a random variable $X$ can be written as:

$$\mu_{X_p} = \mu = E\{X^P\} = \int_{-\infty}^{\infty} x^p f_X(x)dx$$

(3.3)

whereas

$$\eta_{X_p} = \eta = E\{(X - \mu_X)^P\} = \int_{-\infty}^{\infty} (x - \mu_X)^p f_X(x)dx$$

(3.4)

is termed the $p^{th}$ moment. In equation (3.3) and (3.4), ‘$E$’ is the expected value operator which defined by

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

(3.5)

In order to be able to give meaning to concept of stochastic process the joint distribution function of a sequence of random variables $X_1, X_2, \ldots, X_n$ should be introduced:

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_j \leq x_j, \ j = 1,2,\ldots n)$$

(3.6)

A sequence of random variables $X_1, X_2, \ldots, X_n$ may be describing the evolution of a stochastic system over discrete instants of time $t_1, t_2, \ldots, t_n : X(t_1), X(t_2), \ldots, X_n \ldots$. This is called stochastic process. The totality of joint distribution functions $F_{X_{j_1}, X_{j_2}, \ldots, X_{j_k}}$ ($j_k = 1,2,\ldots$ and $k = 1,2,\ldots$) is termed the probability law of the stochastic process. Hereafter, these distributions will be written as $F_{t_{j_1}, t_{j_2}, \ldots, t_{j_k}}$ to emphasize the role of time instants. When all the join distribution functions are Gaussian we call the process a Gaussian process.

A stochastic process can be considered as stationary or nonstationary process. Moreover, a process is strictly stationary if its joint distribution functions are all invariant under time displacement:

$$F_{t_1+\Delta t, t_2+\Delta t, \ldots, t_n+\Delta t} = F_{t_1, t_2, \ldots, t_n}, \ n = 1,2,3,\ldots$$

(3.7)

When the mean $\mu_X = \mu_1 = \mu$ and the variance $\sigma_X^2 = \eta_2$ are constant and the covariance function satisfies $C_{XX}(t_j, t_k) = E\{(X_j - \mu)(X_k - \mu)\} = C_{XX}(t_j - t_k)$, the process is termed as wide-sense stationary. The latter form of stationarity is a weaker stationarity condition with respect to the first-order and the second-order moment is required.

Furthermore, in engineering practices the term ergodicity is quite common in context to stationary process. A stationary process $X(t)$ (with time taking values in $\mathbb{R}$) with mean $\mu$ and
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autocorrelation function $R_{XX}(\tau) = E \{ X(t) X(t + \tau) \}$ is ergodic if for every realization $x_k(t)$ of the process $X(t)$ hold following relation:

$$\mu = \bar{x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_k(t) dt$$

(3.8)

$$R_{XX}(\tau) = R_{x_k x_k}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_k(t) x_k(t + \tau) dt \quad \forall \tau$$

(3.9)

Thus, an ergodic process is characterized by the fact that the time averages of realization equal to the ensemble averages. When both stationarity and ergodicity hold, the value of Gaussian process on a certain time instant is a random variable with a Gaussian probability density function. Furthermore, this probability density function is same for all time instants.

Besides the above information, a stochastic process can also be analyzed in frequency domain. The power spectral density function $S_{XX}(\omega)$ of stochastic process $X(t)$ can be defined as Fourier transform of the autocorrelation function:

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau,$$

(3.10)

3.2 White Noise

In random Vibrations, white noise is extensively used to model excitations with a broad-banded frequency spectrum. The name ‘white noise’ comes from the fact that its average power is uniformly distributed in frequency, which is a characteristic of white light.

This it has a constant power spectral density. This means that all frequencies contribute equal amount of energy to stochastic process up to infinite frequencies. Consequently, a white noise process has infinite variance. The fact that the infinitely high frequencies contribute to the process implies that it has zero memory.

The stationary covariance function $C_{\xi\xi}(\tau)$ of a white noise process (with zero mean $\mu_{\xi}$), therefore, is a constant multiple of the Dirac delta function $\delta(t)$:

$$C_{\xi\xi}(t) = E \{ W(t) W(t + \tau) \} = R_{\xi\xi}(\tau) = D \delta(t)$$

(3.11)

where D is termed for intensity of the white noise. This implies that the two values of a realization of a white noise process are uncorrelated, no matter how close they are in time.

Combining equation (3.9) and (3.10) yields the power spectral density of a white noise:

$$S_{\xi\xi}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{\xi\xi}(\tau) e^{-i\omega\tau} d\tau = \frac{D}{2\pi} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = \frac{D}{2\pi} S_0$$

(3.12)
Due to the characteristic mentions above, white noise cannot be a stochastic process in usual sense, but must be interpreted in the sense of generalized functions (like the Dirac delta function).

For obvious reasons, white noise process cannot exist in practice. However, white noise can be used to model broad-banded processes. Moreover, using filtered white noise (colored noise) one can approximate processes with wide variety of spectral characteristics.

When the excitation is a white noise process, the response generally is a Markov process. [Kloeden and Platen, 1992]. So, if a system is nonlinear or if multiplicative random excitations are present, or both, then, mathematically exact solution is not always obtainable. When such an exact solution is obtained, it is usually based on the assumption that system response is a Markov stochastic process or related to Markov process in some sense.

3.3 Markov Process

A stochastic process $X(t)$ is said to be a scalar Markov process if it has the property

\[
\begin{align*}
\text{Prob}\{X(t_n \leq x_n) | X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1\} \\
= \text{Prob}\{X(t_n) \leq x_n | X(t_{n-1}) = x_{n-1}\} & \quad t_n > t_{n-1} > \cdots > t_1
\end{align*}
\]

(3.13)

Where $\text{Prob}[\ . \ ]$ denotes the probability of an event, and where the statement following a vertical specifies certain conditions under which such a probability is defined. In the present case, the conditions are known values of $X(t)$ at earlier time instants, $t_1, \ldots, t_{n-1}$.

A sufficient condition for $X(t)$ to be Markov process is that its increment in any two nonoverlapping time intervals are independent.

For stochastic dynamics applications, a Markov process usually assumed to be continuously valued, and its time parameter $t$ is defined on a continuous space. If the values of the parameter $t$ are discrete, it is known as Markov series. If both the space and time parameter are discrete, it is called Markov chain.

The conditional probability, $\text{Prob}\{X(t) \leq x | X(t_0) = x_0\}$ of a Markov process $X(t)$ is called the transition probability distribution function. A Markov process is completely characterized by its probability distribution and its probability distribution at an initial time. If the transition probability distribution function of a Markov process is differentiable, its transition probability density function can be obtained, which is often easier to deal with. Transition probability density function can be obtained by:
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\[ q(x,t|x_0,t_0) = \frac{\partial}{\partial x} \text{Prob} \{ X(t) \leq x | X(t_0) = x_0 \} \quad (3.14) \]

The concept of scalar Markov process is readily generalizable to a vector Markov process. Thus \( X(t) = \{X_1(t), X_2(t), \ldots, X_m(t)\}' \) is an m dimensional Markov vector if it has the property of

\[
\text{Prob}\left[ \bigcap_{j=1}^{m} \{ X_j(t_n) \leq x_j \} | X(t_{n-1}) = y_{n-1}, \ldots, X(t_i) = y_i \right] \\
= \text{Prob}\left[ \bigcap_{j=1}^{m} \{ X_j(t_n) \leq x_j \} | X(t_{n-1}) = y_{n-1} \right] \quad t_n \succ t_{n-1} \succ \ldots \succ t_i \quad (3.15)
\]

Where \( \bigcap \) denotes the joint occurrence of multiple events. A sufficient condition for vectorially valued stochastic process to be a Markov vector is that its vectorial increment be independent in nonoverlapping time intervals. The transition probability density of a Vector Markov process is a generalization of

\[
q(x,t|x_0,t_0) = \frac{\partial^m}{\partial x_1 \ldots \partial x_m} \text{Prob}\left[ \bigcap_{j=1}^{m} \{ X_j(t) \leq x_j \} | X(t_0) = x_0 \right] \quad (3.16)
\]

The higher order probability densities, describing the behaviour of a Markov process at several instants of time, can be constructed from initial probability density and transition probability density as follows;

\[
p(x_1,t_1; x_2, t_2; \ldots; x_n, t_n) = q(x_n,t_n|x_{n-1}, t_{n-1})q(x_{n-1}, t_{n-1}|x_{n-2}, t_{n-2}) \ldots \\
q(x_2, t_2|x_1, t_1)p(x_1) \quad t_1 < t_2 < \cdots < t_n \quad (3.17)
\]

3.4 The Fokker-Planck Equation

In many practical applications, the initial state of Markov process is known. Then the transition probability density alone characterizes completely the stochastic process. If only stationary behaviour of process is of interest, then its probability density can be obtained from

\[
\frac{\partial}{\partial t} q + \frac{\partial}{\partial x_j} \left( A_jq \right) - \frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} \left( B_{jk}q \right) = 0 \quad (3.18)
\]

The physical implication of above equation, as the duration of transition time increases, the effect of initial condition \( X(t_0) = x_0 \) diminishes, and the conditional probability density tends to unconditional probability density. This unconditional probability density describes the stochastic process at an arbitrary instant of time after it attains the stationary state and it must be independent of time.
Furthermore, at stationary state, the transition probability density must be function of net transition time; \( q(x, t|x_0, t_0) \) must be a function of \( t - t_0 \) and higher order probabilities can be obtained as

\[
p(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = q(x_n, t_n - t_{n-1} | x_{n-1}) q(x_{n-1}, t_{n-1} - t_{n-2} | x_{n-2}) \ldots \]

\[
\times q(x_2, t_2 - t_1 | x_1) p(x_1) \quad t_1 < t_2 < \cdots < t_n \tag{3.19}
\]

where \( q(x, \tau | x') \) is abbreviation for \( q(x, \tau | x', t) \). Again transition probability density alone characterizes a stationary Markov process, since the additional requirement of the unconditional one time probability density is obtainable from equation (3.19).

\[
p(x_2, t_2; y, t | x_1, t_1) = q(x_2, t_2 | y, t) q(y, t | x_1, t_1) \quad t_1 < t < t_2 \tag{3.20a}
\]

Integrating equation (3.17) with respect to \( y \), we obtain

\[
q(x_2, t_2 | x_1, t_1) = \int q(x_2, t_2 | y, t) q(y, t | x_1, t_1) dy \tag{3.20b}
\]

where \( dy \) is an abbreviation for \( dy_1, dy_2 \ldots dy_m \) and the integration is to cover the entire \( m \)-dimensional domain. The above equation is known as Chapman-Kolmogorov-Smoluchowski equation.

The equation (3.21a) is governs the transition probability density of Markov process. The integral of this equation may be converted into an equivalent differential equation, called Fokker-Planck equation. If the Markov process is also a Gaussian process, the by using the Gaussian property that the higher order moments can be expressed as product of first and second moment. [Lin et al., 1967].

Then the Fokker-Planck equation equation can be given as

\[
\frac{\partial}{\partial t} q + \frac{\partial}{\partial x_j} \left( A_j q \right) - \frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} \left( B_{jk} q \right) = 0 \tag{3.21a}
\]

Moreover if the stationary state of a Markov process exists, then the stationary probability density \( p(x) \) is the limit of the transition probability density. This stationary probability density satisfies a reduced Fokker-Planck equation without the time derivative term:

\[
\frac{\partial}{\partial x_j} \left( A_j p \right) - \frac{1}{2} \frac{\partial^2}{\partial x_j \partial x_k} \left( B_{jk} p \right) = 0 \tag{3.21b}
\]

Equation (3.21b) is called reduced Fokker-Planck equation. Furthermore equation (3.22b) can also be written as

\[
\frac{\partial}{\partial x_j} F_j = 0
\]
where $G_j$ is the probability flow in the jth direction when Markov process is in the state of stationarity; that is,

$$G_j = A_j p - \frac{1}{2} \frac{\partial}{\partial x_k} (B_{jk} p)$$

(3.21c)

The equation (3.21c) is analogous to the continuity equation in fluid mechanics, indicating conservation of mass in fluid flow. Similarly equation (3.21c) may be interpreted as conservation of probability, and $G_j$ as the $j^{th}$ component of the probability flow vector.

within the domain in which Markov process is defined, there may exist certain points possessing one of the following two properties:

1. All the second derivate moments $B_{jk}$ must vanish.
2. At least one first derivate moment $a_j$ becomes unbounded.
4 EXACT RESPONSE METHODS FOR RANDOMLY EXCITED SYSTEMS

Mathematically exact solution of randomly excited nonlinear system is difficult to obtain. The possibility of solution does exist, however when random excitations are independent at any two instant of time, in which case the system response, represented as a vector in a state space, is Markov vector. The probability density of a Markov vector is governed by parabolic partial differential equation called the Fokker-Planck equation which has been derived in previous chapter.

Still the full solution to a Fokker-Planck equation, which shows how the probability structure evolves with time, know for very special first order systems for which system response is Markov scalar process.

For higher order non-linear systems, solution has been obtained in few cases [Dimentberg, 1982], [Crandall, 1962], [Lin et al., 1987] for reduced Fokker-Planck equation without the time derivative term. The unknown in a reduced Fokker-Planck equation is the probability density of system response when it reaches to state of statistical stationarity. Of course, a stationary response exists only if several necessary conditions are met, including:

1. Every random excitation is stationary process,
2. System parameters are time invariant, and
3. Some energy dissipation mechanism exists in the system such that the energy input from the random excitations is balanced statistically by energy output from dissipation.

A stationary solution, if obtainable, is very useful. It is needed for computing the statistical averages for some system response variable to cross over specified boundaries, which are useful or reliability assessment. Stationary response also provides information on the possibility of instability and/or bifurcation of the response.
One of the first to obtain an exact, stationary solution of the FPK-equation for systems with a nonlinear restoring force and linear damping under external random excitations was Kramer [1940], Caughey [1964], and Caughey and Ma [1983] extended the solutions to include certain types of nonlinear damping, but still restricted to external excitations. Yong and Lin [1987], Cai and Lin [1988a] and Cai and Lin [1988b] developed a systematic procedure to obtain exact stationary response solutions for either external or parametric excitations, or both. The class of nonlinear systems, for which this procedure is applicable, is termed the class of generalized stationary potential, and is claimed to be the broadest class of solvable, nonlinear, stochastic systems up to that date. The method is said to be also applicable to MDOF systems. However, the class of generalized stationary potential is too narrow with respect to the class of systems that we strive to cover.

4.1 Stationary Potential

The concepts of stationary potential and detailed balance are introduced to provide the needed physical insight for the development of the systematic procedure for obtaining exact stationary solutions.

The stationary potential terms was first introduced by Stratonovich [1963]. According to that, $X(t)$ be the one dimensional scalar Markov process, for which reduced Fokker-Planck equation can be written as

$$\frac{dG}{dx} = \frac{d}{dx} \left( A(x)p(x) - \frac{1}{2} \frac{d}{dx} [B(x)p(x)] \right) = 0$$

(4.1)

Integrating equation (5.1), which yields

$$G = A(x)p(x) - \frac{1}{2} \frac{d}{dx} [A(x)p(x)] = G_c = constant$$

(4.2)

Since the probability flow vanishes at boundaries, and then the term $G_c = 0$ must vanish everywhere. So solution of equation (5.1) yields

$$p(x) = C e^{-\phi(x)}$$

(4.3)

where $C$ is the normalization constant, and

$$\phi(x) = \ln B - 2 \int \frac{A}{B} dx$$

(4.4)

Function $\phi$ is called probability potential. The exponential form of solution of equation (4.3) guarantees that $p(x)$ is nonnegative for a positive normalization constant $C$.

The term stationary potential refers to a special case in which the probability flow does vanish everywhere, not only at the boundaries.

$$G_i = A(x)p(x) - \frac{1}{2} \frac{\partial}{\partial x_i} [B_{ij}p(x)] = 0$$

(4.5)
For every $i$, substituting equation (5.3) in equation (5.5), one obtain

$$B_{ij} \frac{\partial \phi}{\partial x_j} = \frac{\partial}{\partial x_j} B_{ij} - 2A_i \tag{4.6}$$

These are n equations for $\phi$. The stochastic system is said to belong to the class of stationary potential if a consistent $\phi$ can be found to satisfy all these equations.

### 4.2 Generalized stationary potential

For obtaining, both the first and second derivate moment are split as follows

$$A_i = A_i^{(1)} + A_i^{(2)} \tag{4.7}$$

$$B_{ij} = B_{ij}^{(1)} + B_{ij}^{(2)} \tag{4.8}$$

Where $A_i^{(1)}$ and $A_i^{(2)}$ not restricted to irreversible and irreversible and subscript $(i)$ and $(j)$ does not imply a summation.

Now $p$ can be expressed on terms of potential function $\phi$ as follows;

$$A_i^{(1)}(x) = \frac{\partial}{\partial x_j} B_{ij}^{(1)}(x) - B_{ij}^{(1)}(x) \frac{\partial \phi}{\partial x_j} \tag{4.9}$$

and

$$\frac{\partial}{\partial x_i} A_i^{(2)}(x) = A_i^{(2)}(x) \frac{\partial \phi}{\partial x_i} \tag{4.10}$$

respectively. There are total of $n + 1$ equations for $\phi$ in set of equations (4.7) and (4.8). problem is solvable if a consistent $\phi$ function can be obtained from these equations. Such a solvable system is said to belong to the class of generalized stationary potential. So a system belonging to generalized stationary potential, its stationary probability density can be given by

$$p(x_1, x_2) = C \exp(-\phi) = C \lambda \exp(-\phi_0) \tag{4.11}$$

where $C$ is normalization constant $\phi_0(\lambda)$ is an arbitrary function of $\lambda$.

### 4.3 Closing Remarks

Mathematically exact solutions for randomly excited non-linear systems are difficult to obtain but if a stationary solution obtainable is still very useful. Herein, present method, the strong restriction that the oscillator must belongs to certain specific solvable class of problems for which solution is obtainable makes method for limited applications.
5 APPROXIMATE RESPONSE METHODS FOR RANDOMLY EXCITED SYSTEMS

When a multidimensional nonlinear system is subjected to both parametric and external random excitations of Gaussian white noises, the reduced Fokker-Planck equation can be solved in closed form only with certain highly restrictive relation between the system parameters and the spectral densities of the excitations. Under practical condition such restrictive requirement rarely met. Therefore approximate solution techniques are generally needed.

The approach to the problem of determining the time-dependent probability density function is to solve the FPK-equation by numerical means. A simple and efficient numerical scheme can be formulated by employing the random walk analogue Roberts [1981]. This method can only be applied to 1-dimensional FPK equations and is, therefore, not applicable to MDOF systems.

The numerical integration of the FPK-equation for MDOF systems quickly becomes very cumbersome, because of the high dimension of the probability space that is to be discretized. Summarizing, it can be concluded, that the Fokker-Planck equation method is not suitable to be applied to a wide class of practical MDOF systems. An even more important shortcoming of the method is that is does not provide information on the power spectral density of the response.

The most frequently used approximation scheme is the equivalent linearization procedure. [Cauchy, 1959a, 1959b],[Cai et al., 1990],[Spanos, 1981] in which the original system is replaced by an equivalent linear system. The parameter of the replacement linear system is determined using the statistical criterion of least-mean-square difference.
5.1 Perturbation Method

In the classical perturbation method, see Crandall [1963], Lin [1967] and Nayfeh [1981], the basic idea is to expand the solution to the nonlinear set of equations in terms of a small scaling parameter, which characterizes the magnitude of the nonlinear terms in these equations. The first term in the expansion is simply the linear response, which is the response when all the nonlinearities in the system are removed.

The subsequent terms express the influence of the nonlinearity. As with perturbation in general, the calculations are usually lengthy and rapidly become more tedious as the order of the scaling parameter increases. In practice, results are usually obtained only to the first order in the scaling parameter. The method is, therefore, only valid for small perturbations. Consequently, the perturbation method can only be applied effectively when weakly nonlinear systems are considered.

The Perturbation method can be extended to mdf systems. By expanding the response in powers of perturbation parameter, the equation can be reduced to hierarchy of a system of linear differential equation that can be solved sequentially by linear system theory.

5.2 The Fokker-Planck-Kolmogorov Equation Method

The Fokker-Planck (FPK) equation method can provide information on the stationary (or non-stationary) unnormalized probability density function of the response of a dynamic system. The FPK equation is a partial differential equation for the probability density function of the response incorporating partial derivatives to this response (and time, in case of a non-stationary probability density function). This equation should be solved under appropriate boundary (and initial) conditions. For a more detailed discussion of the Fokker-Planck equation method refer to Caughey [1971] and Dimentberg [1982]. One could distinguish between methods providing exact solutions and methods providing approximate solutions of the FPK-equation. The method is described in detail in chapter 6.

5.3 Stochastic Averaging

The method of stochastic averaging has proven to be a very useful tool for deriving approximate solutions to problems involving the vibration of weakly damped systems to broad-band random excitation. It was proposed initially by Stratonovich [1963].

For solving problems concerning noise-excited, dynamic systems, Subsequently, Stratonovich's method was justified and interpreted rigorously by Khasminskii [1966] and
Papanicolaou et al. [1974]. For reviews on the stochastic averaging method readers can refer to Roberts and Spanos [1986] and Zhu [1988].

The basic idea is to use the Markov approximation for the response, so that the probability density function can be described by the FPK-equation. The stochastic averaging method was devised to obtain the coefficient functions in this partial differential equation. For a re derivation of the formulas required for the application of Stratonovich's stochastic averaging method, see Lin [1986].

The goal of the method is to simplify the FPK-equation, or even reduce the dimension of the FPK-equation [Zhu, 1988]. Thus, by using stochastic averaging methods, the difficulties in solving the FPK-equation are relieved and the range of application of the FPK equation method can be extended. In random vibration studies, the stochastic averaging method has been applied principally to systems with one degree of freedom.

Often, the equation of motion is a one-dimensional, second order differential equation. Then, the stochastic averaging method enables the basic two-dimensional (in mechanical systems often displacement and velocity) Markov process governing the response to be replaced, approximately, by a one-dimensional Markov process governing an envelope amplitude process.

The appropriate FPK-equation for the envelope amplitude can be easily solved analytically to yield simple expressions for the stationary probability distribution of the amplitude process. By considering an associate phase process, approximate analytical expressions for the joint distribution of the response displacement and velocity can be derived. The reduction in dimension of the governing FPK-equation, from two to one, also considerably simplifies the computation of non-stationary or transient solutions. In general, the application of the stochastic averaging method is constrained to weakly damped systems.

For the application to systems with high damping, the reader can referred to Sri Namachchivaya and Lin [1988]. Furthermore, the application of the method is generally limited to SDOF systems. The application to MDOF systems is very limited due to the difficulties in solving a multi-dimensional FPK equation. However, it should be noted that externally as well as parametrically excited systems can be examined.

Furthermore, the standard stochastic averaging method is not particularly useful for examining the effect of nonlinear 'restoring forces'. Namely, the effect of these forces on the
probability density function vanishes after averaging. However, in reality the nonlinear 'restoring forces' can markedly affect the probability distribution of the response. An approach to tackle this problem is 'stochastic averaging of the energy envelope' [Roberts and Spanos, 1986], in which the total energy of the system is approximated by a one-dimensional Markov process. Its probability distribution can then be described by the FPK-equation. This procedure is also only applicable to SDOF systems. Summarizing, the stochastic averaging method is not suitable to tackle problems incorporating MDOF systems with nonlinearities of a general form. Moreover, the method provides no information in the frequency domain incorporating MDOF with nonlinearities of a general form.

5.4 Closure Techniques

Closure techniques are also based on the Markov process assumption. A very important tool for the analysis of Markov processes is the Ito stochastic calculus [Ito, 1944]. Using the Ito stochastic calculus, the so-called moment equations can be derived. Generally, the moment equations are a set of differential equations in the statistical moments of the response. For stationary problems, the set of differential equations reduces to a set of algebraic equations.

It is well known that the response of linear, time-invariant systems subject to Gaussian excitations will generally be Gaussian too [Schetzen, 1980]. As a consequence, the (Gaussian) probability density function can be described using only the first-order and second-order statistical moments (corresponding to the mean and the mean square of the response). In this case, these moments can be determined by solving two algebraic moment equations.

However, for nonlinear systems the response to Gaussian excitations is generally non-Gaussian [Wiener, 1942]. The corresponding, non-Gaussian probability density function cannot be described using merely the first two statistical moments; higher order moments become important. These statistical moments are, then, governed by an infinite hierarchy of coupled equations. Therefore, some form of a closure scheme has to be applied in order to make the set of moment equations solvable.

In stochastic dynamics the term 'closure' refers to a procedure, by which an infinite hierarchy of equations governing the statistical moments of random quantities is truncated and the values of lower-order moments are computed approximately. One can distinguish between 'Gaussian' and 'non-Gaussian' closure.

5.4.1 Gaussian Closure
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The simplest closure scheme is the Gaussian closure, in which higher moments are expressed in terms of the first-order and second-order moments as if the random processes involved were normally distributed. In order to express these moments of higher order (higher than two) in terms of the first-order and second-order moments a 'cumulant neglect closure' (CNC) scheme is used, see Wu and Lin [1984]. In the Gaussian CNC, the closure of the infinite hierarchy of moment equations is achieved by setting the third-order and higher-order cumulants (also called semi-invariants) to zero; see Ibrahim [1985] or Nikias and Petropulu [1993] for a mathematical definition of the cumulants. The result of the application of the Gaussian CNC is a closed set of nonlinear moment equations concerning the first-order and second-order moments. It should be noted that this can be implemented for MDOF systems.

Moreover, stationary as well as non-stationary problems can be tackled. In the stationary case, the remaining moment equations are algebraic. When non-stationary moments are to be computed a set of differential equations will have to be solved by integration. The approximate moments can be used to form a Gaussian probability density function. Gaussian closure can yield satisfactory results for systems with weak nonlinearities. However, the response of strongly nonlinear systems subject to Gaussian excitations will generally be non-Gaussian, and can, therefore, not be described using only first two statistical moments.

5.4.1 Non-Gaussian Closure

To overcome the shortcomings of Gaussian closure, non-Gaussian closure schemes were proposed by Wu and Lin [1984] and Crandall [1985]. In the non-Gaussian closure, non-Gaussian features of the response are taken into account. Mathematically speaking, the Gaussian closure can be generalized by successive inclusions of additional terms, which describe the non-Gaussian features in greater and greater detail. Of course, the complexity of the closed set of moment equations to be solved becomes greater. This becomes even more evident when the system has discontinuous nonlinearities. Two approaches can be distinguished. In the first approach, the CNC is applied. Since the third-order and higher-order cumulants of Gaussian random variables are zero, successive improvements over the Gaussian closure can be obtained by including additionally the third-order, fourth-order, and fifth-order cumulants and so on. For example, one could set the fifth-order and higher-order cumulants to zero.
Chapter 5. Approximate Response Methods for Randomly Excited System

The result is a closed set of nonlinear equations concerning the first-order, second-order, third order and fourth-order moments. This approach is applicable to MDOF systems with strong nonlinearities. However, the nonlinearities have to be of a polynomial kind. Furthermore, external as well as parametric excitations can be treated and stationary as well as non-stationary problems can be tackled.

In the second approach, the unknown probability density function of the response is approximated by a truncated Gram-Charlier or Edgeworth series [Crandall, 1985; Ibrahim et al., 1985], in which the first term is the Gaussian distribution. The coefficients of the infinite series are then determined using the dynamic equations of motion of the system. Following this approach, problems which include nonlinearities of higher complexity can be treated as well. However, MDOF systems and non-stationary problems are difficult to treat, due to the difficulties in defining appropriate probability density functions for these cases.

It can be concluded that non-Gaussian CNC can be appropriate to tackle problems concerning MDOF systems and strong nonlinearities. However, the application to systems with discontinuous nonlinearities implies the solution of rather complex nonlinear equations. Furthermore, in Sun and Hsu [1987] it was stated that it might occur that the validity of the results, provided by the non-Gaussian CNC, is restricted to specific areas of parameter values of the equations of motion.

In such cases, non-Gaussian CNC would provide erroneous results in certain parameter areas. However, in many cases, the extension to the non-Gaussian closure does lead to an improvement of accuracy of the moments. The most important drawback of the closure techniques, with respect to the aims of this research, is the fact that this method does not provide information on the power spectral density of the response.

For demonstrating numerical result using closure technique a duffing oscillator studied by Wu and Lin 1884 under additive random excitation yield better result for higher order of cumulants.

The equation of oscillator can be given by-

\[ \ddot{X} + \eta \dot{X} + X + \varepsilon X^3 = \sqrt{\eta}W(t) \]  \hspace{1cm} (5.1)

where \( \varepsilon \) is a parameter representing the degree of non-linearity and \( W(t) \) is a Gaussian white noise with a correlation function

\[ E[W(t)W(t + \tau)] = 2\delta(\tau) \]  \hspace{1cm} (5.2)
Chapter 5. Approximate Response Methods for Randomly Excited System

The stationary probability of the displacement $X = X_1$ and the velocity $\dot{X} = X_2$ is known to be [Dimentberg, 1982]

$$p(x_1, x_2) = C \exp \left( -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4 \right)$$ (5.3)

where $C$ is the normalization constant.

The mean square displacement can found out by

$$E[X_1^2] = \int_{-\infty}^{\infty} x_1^2 p(x_1, x_2) \, dx_1$$ (5.4)

For a small value of $\varepsilon$, analysis results following asymptotic expansions:

From Gaussian closure:

$$E[X_1^2] \approx 1 - 3\varepsilon + 18\varepsilon^3 - \cdots$$ (5.5)

From the fourth order cumulant-neglect-closure:

$$E[X_1^2] \approx 1 - 3\varepsilon + 24\varepsilon^2 - 297\varepsilon^3 + 453\varepsilon^4 - \cdots$$ (5.6)

From the sixth order cumulant-neglect-closure:

$$E[X_1^2] \approx 1 - 3\varepsilon + 24\varepsilon^2 - 297\varepsilon^3 + 4986\varepsilon^4 - 100,278\varepsilon^5 + \cdots$$ (5.7)

From exact solution:

$$E[X_1^2] \approx 1 - 3\varepsilon + 24\varepsilon^2 - 297\varepsilon^3 + 4986\varepsilon^4 - 100,278\varepsilon^5 + \cdots$$ (5.8)

Figure (5.1) shows the cumulant-neglect procedure does have a tendency to converge to the exact solution for all values of $\varepsilon$. But the cumulant-neglect-closure scheme may lead to larger errors when the system response is near a stability boundary.

![Figure (5.1) stationary mean square response of duffing oscillator for Gaussian closure, 4th, 6th cumulant-neglect closure and exact solution](image-url)
5.5 Linearization Methods

A natural method of attacking nonlinear problems is to replace the governing set of nonlinear differential equations by an equivalent set of linear differential equations; the difference between the sets being minimized in some appropriate sense.

The stochastic linearization technique can be considered to be an extension of the equivalent linearization method for the treatment of nonlinear systems under deterministic excitations [Krylov and Bogoliubov, 1943]. Caughey [1963b] was one of the first to apply the stochastic linearization technique to randomly excited nonlinear systems.

The basic idea of the statistical linearization approach is to replace the original nonlinear system by a linear one. This is done in such a way that the difference between the two systems is minimized in some statistical sense. In this way, the parameters of the linearized system are determined. The response of the nonlinear system is approximated by the response of the equivalent linear system. So, the unknown statistics of the response are evaluated approximating the response as a Gaussian process, when the excitation is assumed to be Gaussian. Recently, Roberts and Spanos [1990] provided a comprehensive account on statistical linearization.

The use of the Gaussian approximation for the response suggests that stochastic linearization is very close to Gaussian closure. For externally excited systems, the two approaches provide the same results. However, then parametric excitations are applied, different approaches can be used, which do not all show equivalence to Gaussian closure: linearization of the equations of motion; linearization of the Itô equation of the system; linearization applied to the coefficients of Itô’s differential rule [Falsone, 1992]. In Falsone [1992], it is shown that only the third approach is equivalent to Gaussian closure in case of parametrically excited systems. The methods are extended for application to MDOF systems; see Roberts and Spanos [1990] and Falsone [1992].

A feature which distinct statistical linearization from all the methods, discussed earlier (excluding numerical integration), is its capability to provide approximate information on the power spectral density of the response very easily. Moreover, the response statistics can be computed analytically, once the actual linearization has been performed. Consequently, the method is computationally very efficient compared to numerical integration.
The linearization approach can be applied to both white and non-white inputs. Furthermore, Zhu et al. [1993] have investigated a Duffing oscillator subjected to narrow-band excitation by means of simulation. It is well-known, that it is possible for nonlinear systems subjected to sinusoidal excitation to exhibit multiple stable solutions (depending on the initial conditions) in certain parameter areas.

For narrow-band stochastic excitation, it is shown in Zhu et al. [1993] that for each combination of the parameters all the statistics of the stationary response are unique and independent of the initial conditions. However, in a certain domain of the parameter space there are two more probable motions in the stationary response and jumps between those more probable motions may occur. This phenomenon only occurs when the frequency band of the excitation is small enough.

It was shown by Richard and Anand [1983] that the statistics of these more probable motions can be computed by means of the linearization technique. The 'multiple solutions' computed by the linearization technique correspond, to some extent, to the "local" behaviour of sample functions of the response. The response statistics of the total response can only be computed by linearization when it is known how much time the system spends in the two more probable motions. So, it can be concluded that the linearization can be applied to problems with a wide variety of excitation forms.

Furthermore, it can be applied to systems with non-polynomial and discontinuous nonlinearities. The main limitation of the method lies in the fact that it will only provide accurate results for (very) weak nonlinearities. So far, linearization of the equations of motion of nonlinear systems has been achieved by replacing the nonlinear terms in the equations of motion by zero-memory linear terms. This is optimal when the input is Gaussian. However, in the case of a nonlinear system the relevant input to the nonlinear term is the response process, which is often distinctly non-Gaussian (this becomes more evident for stronger nonlinearities).

So, conventional linearization is not optimal in these cases. Increased accuracy may be obtained through the introduction of memory into the linear substitution. A relatively straightforward approach to introducing memory has been given by Iyengar [1988]. The resulting linear system is of a higher order than the original nonlinear system. Therefore, the
approach is called higher-order linearization. Iyengar [1988] has shown that, for the case of a Duffing oscillator excited by white noise, a fourth-order equivalent linear system leads to a significant improvement in accuracy, with regard to the mean square of the response.

Moreover, the power spectrum obtained yields an estimate of the power spectrum of the response which shows two peaks, reflecting the existence of sub harmonics in the system. This is in reasonable agreement with digital simulations. However, application to systems with discontinuous nonlinearities has not been investigated yet and problems are expected to occur with regard to the analytical computation of the derivatives of the nonlinear terms, which are needed in this approach.

5.6 Method of Weighted Residual

In equivalent nonlinear methods the original nonlinear system is replaced by an equivalent nonlinear system. The replacing, equivalent, nonlinear system should be a system of which the nonlinear stochastic response can be determined rather easily.

In the method proposed by Cai and Lin [1988a] and Cai et al. [1992], the original nonlinear system is replaced by a nonlinear system belonging to the class of generalized stationary potential. This class of systems is the broadest class of nonlinear systems of which exact solutions for the stationary response can be obtained. In order to choose an appropriate replacement system, in this method, a residual is defined as a measure of the difference between the two systems. This residual is defined as the error in the original FPK-equation, introduced by using the solution of the equivalent system as an approximation for the solution of the original system. The residual is minimized by means of the method of weighted residuals. This results in a set of constraints for obtaining an approximate stationary probability density function. One of the constraints coincides with the criterion of dissipation energy balancing. This criterion implies that the average energy dissipation is the same for the original as it is for the replacement system.

The other constraints are useful to calculate the equivalent conservative force of the equivalent system. When the equivalent system is known, the stationary probability density of this system can be determined and can be used as an approximate solution for the original system. This method can yield results with much higher accuracy than those obtained by statistical linearization. Non-Gaussian properties are covered (partly), because the original
system is not linearized. However, the method is only applicable to SDOF systems and does not provide frequency-domain response information.

Another equivalent nonlinear method is known as partial linearization; see Elishakoff and Cai [1992]. In partial linearization only the damping of the system is linearized. The equation thus obtained is amenable to an exact solution. The equivalent (linear) damping parameter is selected by means of the dissipation energy balancing criterion, see Cai and Lin [1988a] and Cai et al. [1992]. The proposed procedure considerably improves the accuracy of the statistical linearization method and yields simple equations to determine probabilistic characteristics of the system.

The results are less accurate than those obtained by the method described above (of which partial linearization is a special case). However, the computational efforts are reduced. Unfortunately, also the partial linearization method is only applicable to SDOF systems.

The main shortcomings of these methods are, thus, the lack of applicability to MDOF systems and the fact that no frequency-domain information can be obtained. A method that can be applied to MDOF systems and provides information on the power spectral density of the response is equivalent statistical quadratization. The equivalent statistical quadratization method was introduced by Spanos and Donley [1991, 1992] as an extension to the equivalent linearization method. The linearization method often fails to estimate the spectral properties of the response accurately for strongly nonlinear systems. This is sometimes due to the fact that the power spectra of the response of linear systems span only the frequency range of the excitation spectrum. However, significant responses outside this range are possible for nonlinear systems.

Method of weighted residual based on computing directly unknown probability density instead of computing approximate statistical moments. Method is based on the fact that if the exact solution of nonlinear system is not obtainable, then system is replacement by a linear system which is within the solvable class of generalized potential as well close to the original system in some statistical sense. Then the exact solution of the replacement system may be considered as an approximate solution for the replaced original system. [Cai et al. 1992]

Consider a nonlinear system

\[
\frac{d}{dt} X_i = F_i(X) + g_{ij}(X)W_j(t) \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m
\]  

(5.9)
where $W_j(t)$ are Gaussian white noises. The reduced Fokker-Planck equation associated with concern nonlinear system can be written as

$$\frac{\partial}{\partial x_i} [(F_i + \pi K_{ts} g_{rs} \frac{\partial g_{ij}}{\partial x_r})\tilde{p}] - \pi K_{ts} \frac{\partial^2}{\partial x_i \partial x_j} (g_{ij} g_{js}\tilde{p}) = 0 \quad (5.10)$$

where $K_h$ is the cross spectral density of $W_i(t)$ and $W_j(t)$ and $\tilde{p}$ is the stationary probability density of the response vector $X$ and the replaced is governed by

$$\frac{d}{dt} X_i = f_i(X) = g_{ij}(X) W_j(t) \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \quad (5.11)$$

and $p(x)$ be the exact probability density of its response; that is satisfy the reduced Fokker-Planck equation as follow

$$\frac{\partial}{\partial x_i} [(F_i + \pi K_{ts} g_{rs} \frac{\partial g_{ij}}{\partial x_r})\tilde{p}] - \pi K_{ts} \frac{\partial^2}{\partial x_i \partial x_j} (g_{ij} g_{js}\tilde{p}) = 0 \quad (5.12)$$

The residual error in original system’s reduced Fokker-Planck equation using exact probability of replaced system

$$\delta = \frac{\partial}{\partial x_i} [(F_i + \pi K_{ts} g_{rs} \frac{\partial g_{ij}}{\partial x_r})\tilde{p}] - \pi K_{ts} \frac{\partial^2}{\partial x_i \partial x_j} (g_{ij} g_{js}\tilde{p}) = 0 \quad (5.13)$$

Error associated can be obtained by subtracting equation (5.13) with equation (5.12)

$$\delta = \frac{\partial}{\partial x_i} [(F_i(x) - f_i(x))p(x)] \quad (5.14)$$

The residual $\delta$, as a function of $x$, is a measure of error of approximate solution $p(x)$ from the unknown true solution $\tilde{p}(x)$.

The method of weighted residual is global minimization scheme, in which function $f_i$, hence $p(x)$ are chosen such that the integral

$$\Delta_M = \int M(x)\delta dx = \int M(x) \frac{\partial}{\partial x_i} [(F_i(x) - f_i(x))p(x)] dx = 0 \quad (5.15)$$

For some selected weighting function $M(x)$, integrating by part of equation (5.15),

$$\Delta_M = E \left\{ [F_i(X) - f_i(X)] \frac{\partial M(x)}{\partial x_i} \right\} = 0 \quad (5.16)$$

Where $E[.]$ denotes the ensemble average with respected to the probability density $p(x)$. by appropriate selecting the set of $M$, one obtain constraints of equation (5.16). The function $f_i$ for the replacement system can be determined from these constraints. So obviously, the required number of constraints hence required number of weighting function depends upon the parameters to be determined in functions of $f_i$. 

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Furthermore assuming the weighting function is chosen in form of $M(X) = X_{1}^{k_{1}}, X_{2}^{k_{2}}, ..., X_{n}^{k_{n}}$, where $k_{1}, k_{2}, ..., k_{n}$ are positive integers. The equation of $E[M]$ for each set of $k_{1}, k_{2}, ..., k_{n}$ can be obtained as:

$$\frac{d}{dt} E[M] = E[(f_i + \pi K_i g_r \frac{\partial g_{il}}{\partial X_r} \frac{\partial M}{\partial X_i}) + \pi K_i E[g_{il} g_r \frac{\partial^2 M}{\partial X_i \partial X_j}]]$$  \hspace{1cm} (5.17)

After exposure to white noise excitation for sufficiently long time, system attains statistical stationarity, so the expression $\frac{d}{dt} E[M]$ turn to zero and we obtained expression

$$E[(f_i + \pi K_i g_r \frac{\partial g_{il}}{\partial X_r} \frac{\partial M}{\partial X_i}) + \pi K_i E[g_{il} g_r \frac{\partial^2 M}{\partial X_i \partial X_j}]] = 0$$  \hspace{1cm} (5.18)

Substituting equation (5.16) in equation (5.18),

$$E[(F_i + \pi K_i g_r \frac{\partial g_{il}}{\partial X_r} \frac{\partial M}{\partial X_i}) + \pi K_i E[g_{il} g_r \frac{\partial^2 M}{\partial X_i \partial X_j}]] = \Delta_M$$  \hspace{1cm} (5.19)

If constraint (5.16) is satisfied then (5.18) turns to;

$$E[(F_i + \pi K_i g_r \frac{\partial g_{il}}{\partial X_r} \frac{\partial M}{\partial X_i}) + \pi K_i E[g_{il} g_r \frac{\partial^2 M}{\partial X_i \partial X_j}]] = 0$$  \hspace{1cm} (5.20)

Thus the statistical moment computed from the approximate probability density $p(x)$ satisfy equation (5.20)

Let us apply this method to a single-degree-of-freedom second order system governed by

$$\ddot{X} + H(X, \dot{X}) = g_i(X, \dot{X})W_i(t)$$  \hspace{1cm} (5.21)

This system does not belong to category of exactly solvable class of generalized stationary potential. Replacing this system by another system which is belonging to a solvable class of generalized potential. The equation of replaced system can be given as:

$$\ddot{X} + h(X, \dot{X}) = g_i(X, \dot{X})W_i(t)$$  \hspace{1cm} (5.22)

The function $h(X, \dot{X})$ must satisfy the condition

$$\Delta_M = E \left\{ [H(X, \dot{X}) - h(X, \dot{X})] \frac{\partial M(X_1, X_2)}{\partial X_2} \right\} = 0$$  \hspace{1cm} (5.23)

where $X_1 = X$ and $X_2 = \dot{X}$. A set of constraints can be obtained by selecting a set of suitable weighting functions $M(X_1, X_2)$.  

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In principle, the accuracy of approximation is improved by adding more constraints, as long as they are compatible.

The method of weighted residual is applied under three different settings, by selecting the replacement system from increasingly larger classes of systems which are solvable easily.

### 5.6.1 Equivalent Linearization

The method of equivalent linearization is an adequate scheme for weakly nonlinear systems under purely additive random excitations. For Gaussian white excitations, it yields same result as by Gaussian closure scheme. Let us proceed to apply this technique to non-linear sdf oscillator represented by following equation

\[
\ddot{X} + H(X, \dot{X}) = W(t)
\]

(5.24a)

Its linear approximation form can be written as,

\[
\ddot{X} + \beta_e \dot{X} + k_e X = W(t)
\]

(5.24b)

Now applying the constraints given by equation (5.23), we obtain

\[
\begin{bmatrix}
E[X_1^2] & E[X_1 X_2] \\
E[X_1 X_2] & E[X_2^2]
\end{bmatrix}
\begin{bmatrix}
k_e \\
\beta_e
\end{bmatrix}
= \begin{bmatrix}
E[X_1 H(X_1, X_2)] \\
E[X_2 H(X_1, X_2)]
\end{bmatrix}
\]

(5.25)

Equation (3.36) evolves following:

\[
k_e = \frac{E[X_1 H(X_1, X_2)]}{E[X_1^2]}
\]

(5.26a)

\[
\beta_e = \frac{E[X_2 H(X_1, X_2)]}{E[X_2^2]}
\]

(5.26b)

So stationary solution for replaced linear system is well known Gaussian distribution, that is,

\[
p(x_1, x_2) = C \exp\left[-\frac{\beta_e}{2\pi K} (x_2^2 + k_e x_1^2)\right]
\]

(5.27)

where \(K\) is spectral density of white noise. So if function \(H(X_1, X_2)\) is a polynomial of \(X_1\) and \(X_2\), the ensemble averages in equation (5.27a) and (5.27b), can be calculated analytically and \(\beta_e\) and \(k_e\) can be solved in closed form.

### 5.6.2 Partial Linearization

In Section 6.26a, an approximate solution is obtained for non-linear oscillator by seeking its replacement within the class of solvable linear oscillators. But the accuracy can be improved if the replacement system is drawn from a larger class, rather being restricted to linear systems. Here the original system (equation (6.35a)) is replaced by

\[
\ddot{X} + H_1(X, \dot{X}) + g(X) = W(t)
\]

(5.28a)
Where \( H_1(X,\dot{X}) \) represent the damping force, \( g(X) \) represents the conservative force dependent only on the displacement, and \( W(t) \) is a Gaussian white noise. Then a closer replacement system is
\[
\ddot{X} + \beta_e \dot{X} + g(X) = W(t) \quad (5.28b)
\]
In which the conservative force kept unchanged. So replaced system possess following stationary solution
\[
p(x_1, x_2) = C \exp\left\{- \frac{\beta_e}{\pi \kappa} \frac{1}{2} x_2^2 + \int_0^{x_1} g(u) du \right\} \quad (5.29)
\]
and equivalent linear damping force can be determined from
\[
\beta_e = \frac{E[x_2 H_2(x_1, x_2)]}{E[x_2^2]} \quad (5.30)
\]
Equation (5.31) is same as in the case of equivalent linearization, however the \( H_1(X_1, X_2) \) here does not include the non-linear restoring force. So this method is called partial linearization (Elishakoff and Cai 1992), is more accurate than full linearization.

But both of the methods are unsuitable if the multiplicative excitations are present. In figure (5.2), both method were applied on system given by equation \( \ddot{X} + \beta \dot{X} + \gamma \dot{X}^3 + \delta X^3 = W(t), \ \gamma, \delta > 0 \). Figure (5.2) depict the partial linearization is accurate to exact solution than equivalent linearization for different sets of values of parameter \( \beta, \gamma \) and \( \delta \).

### 5.6.3 Energy Dissipation Balancing

The procedure was developed by Cai and Lin [1992]. This is the method that gives more accurate results than other schemes as well it gives good result too when multiplicative excitations are also present. In this procedure, the original non-linear system is replaced by another non-linear system belonging to a class of generalized stationary potential. The criterion for replacement is that the average dissipated energy remains the same for replaced and replacing system.

In applying the dissipation energy method, it is necessary to identify the total effective conservative force for the system, which includes the original conservative force in the equation...
of motion, and possible contribution from so-called Wong and Zakai correction terms. If the total effective conservative force depends only on the displacement, then the same effective force should be used in the replacement system when obtaining the approximate probability density. However, in some cases the forces are functions of velocity or functions of both velocity and displacement can also be conservative, if they do not dissipate energy. So the Energy dissipation balancing procedure is applicable conditionally if the original total conservative force depends only on the system displacement, in this procedure, the selections of both the replacement conservative and non-conservative forces are based on a set of constraints which are derived from method of zero weighted residual. In this section, we will discuss the procedure to apply dissipation energy method in non-linear oscillator. Referring to the system represented by equation (5.22) replaced by system with linear system represented by (5.23). Equation (5.23) belongs to the class of generalized stationary potential if the function h can be expressed in form

$$h(X_1, X_2) = \pi X_2 K_{ij} g_i g_j \phi'(\Lambda) + \pi K_{ij} g_i \frac{\partial g_j}{\partial x_2} + u(X_1)$$  \hspace{1cm} (5.31)$$

where $u(X_1)$ the effective conservative is force and $\Lambda$ is the effective total energy given by

$$\Lambda = \frac{1}{2} X_2^2 + \int u(X_1) \, dX_1$$  \hspace{1cm} (5.32)$$

So the probability density of system response can be given by
\[ p(x_1, x_2) = C \exp(-\phi) \]  \hspace{1cm} (5.33)

Now this form of \( h(X_1, X_2) \) substituted in equation (5.24) and by choosing a set of appropriate weighting functions \( M \), one can obtained a corresponding set of statistical conditions for finding total effective conservative force \( u(X_1) \) and probability potential \( \phi \) for the replacement system. For that substituting function \( M = X_2^2 \) in equation (5.23), that’s leads to

\[ E \left\{ X_2 \left[ H - \pi X_2 K_{ij} g_i g_j \phi'(\Lambda) + \pi K_{ij} g_i \frac{\partial g_j}{\partial x_2} \right] \right\} = 0 \]  \hspace{1cm} (5.34)

Equation (5.34) leads to when substituting equation (5.33),

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -\phi(\lambda) \right] x_2 \left[ H - \pi X_2 K_{ij} g_i g_j \phi'(\Lambda) + \pi K_{ij} g_i \frac{\partial g_j}{\partial x_2} \right] dx_1 dx_2 = 0 \]  \hspace{1cm} (5.35)

Here \( \lambda \) is sum of the kinetic energy \( \frac{1}{2} X_2^2 \) and potential energy

\[ U(x_1) = \int u(x_1) dx_1 \]  \hspace{1cm} (5.36)

The limit of integration is not identified in equation (5.36) since the reference level of potential energy may be chosen arbitrarily. So in general, there always exist a unique point \( x_1 = x_{10} \) where \( U(x_{10}) \) is minimum. If this minimum is considered as zero that is \( U(x_{10}) = 0 \) then,

\[ U'(x_1) = u(x_1) \begin{cases} > 0 & x_1 > x_{10} \\ < 0 & x_1 < x_{10} \end{cases} \]  \hspace{1cm} (5.37)

and that for every \( \lambda > 0 \), the equation \( U(x_1) = \lambda \) has two roots, \( x_1 = \mu_{\lambda_1} \) and \( x_2 = \mu_{\lambda_2} \).

So integration of equation (5.46) on \( x_1 \) and \( x_2 \) and proceed further by putting restrictive sufficient condition that the integration on \( x_1 \) vanishes for every \( \lambda \). This leads to an expression for \( \phi'(\lambda) \). Once \( \phi'(\lambda) \) is known; the stationary approximate probability density can be computed as follows:

\[ p(x_1, x_2) = C \exp\left[ -\lambda \int_0^\lambda \phi'(\nu) d\nu \right] \]  \hspace{1cm} (5.38)

The physical meaning of constraint equation (6.45) is that the average dissipated energy per unit remains the same for the replaced and replacing systems while the strong condition that \( x_1 \) vanishes for every \( \lambda \) requires that the average dissipated energies be identical at every total energy level \( \lambda \). Figure (5.3) shows the good comparison of methods of weighted residual with
Monte-Carlo simulation. The dissipation energy balancing yield result more close to simulation results for different sets of parameters $\beta, \delta$ and $K$.

5.7 Force Analogy Method

As we have seen on previous sections, most of the analytical methods, which discussed have limitations in accuracy, efficiency, and size of the system. On the other hand, the only numerical method available, Monte-Carlo simulation required huge sample size, which required by accuracy, becomes a barrier which difficult to overcome. Therefore an effort to overcome these problems, Wang and Wong [2007], introduce a new method named stochastic force analogy method (SFAM). In the force analogy method, inelastic structural analysis is performed based on varying the structural displacement field instead of varying the stiffness matrices to describe the inelastic behaviours of structures. Doing so, the stiffness matrices remains unchanged as the initial one in the entire process of non-linear dynamic analysis. This characteristic remains in the stochastic procedure. As a result, the efficiency in terms of numerical computation of SFAM has given good results. The SFAM can produce the covariance function and hence a variance function of inelastic structural response subjected to random excitation. These responses include displacement, velocity, inelastic displacement of entire structures, and plastic rotation at all plastic hinges in a moment resisting frame. The SFAM has good agreement with numerically simulated results from Monte-Carlo simulation method.
5.8 Monte-Carlo Simulation

A method for the estimation of the response statistics of randomly excited, nonlinear systems, within any desired confidence level, is based on random computation experiments, popularly known as Monte Carlo simulation, see Rubinstein [1981]. Monte Carlo simulation is generally used to validate the results of other approximation methods. Monte Carlo simulation can provide very accurate results. However, this accuracy can only be obtained at the cost of computational efficiency.

In Monte Carlo simulation, one generates a realization of the excitation. This realization of the excitation is used to compute a realization of the response by numerical integration. Obviously, this approach can be applied to estimate both stationary and non-stationary response statistics. The higher the number of realizations used, the smaller the expected deviation of the obtained numerical values from the theoretical values of the response statistics will be. In order to compute statistical properties of the response accurately, many realizations are required. Therefore, much computationally expensive integration will have to be executed. Obviously, this method is very inefficient from a computational point of view. This problem becomes even more evident for MDOF systems.

However, the method of numerical integration can tackle problems incorporating the entire class of systems and excitations described in section 3.1. Furthermore, both the statistical moments (and probability density function) and the power spectral density can be estimated from the time series of the response that can be computed by numerical integration. This method, thus, answers to all our demands. Unfortunately, these benefits are gained at the cost of computational efficiency. It should be noted, that the necessity of a large number of records can often be eliminated if the interest is confined to stationary response statistics. In engineering practice, it is quite common to assume ergodicity with respect to a specific statistical moment for stationary processes. This assumption allows the determination of this specific ensemble statistical moment by using its temporal counterpart, which is calculated by using a single sample function of the response.

5.9 Closing Remarks

The method of energy dissipation balancing was compared with linearization techniques yield better results but in this method non-linear oscillator is replaced by another non-linear oscillator belonging to class of generalized stationary potential. PDF response is well when the
original oscillator is close to another non-linear oscillator belonging to class of generalized potential. This closeness term is not well defined restricts method of various classes of oscillators.

In many cases of complicated nonlinearities, such as non-as non-differentiable functions or hysteresis or multi-degree-of-systems with polynomial non-linearities, when the standard closure techniques (cumulant or exponential or approximate closure method) fails or lead to very complicated formulas, from a numerical point of view, then statistical or equivalent linearization are the only suitable analytical techniques for response analysis.

Furthermore using Monte-Carlo simulation principally any system and any excitations can be well handled. As a consequence, method is widely applicable. In Monte-Carlo simulation the response statistics are estimated from an infinite number of samples, which can be computed using numerical integration. Consequently, any desired level of accuracy can be obtained. High level of accuracy can, however, only be obtained at the cost of computation efficiency. It can be concluded that the Monte-Carlo simulation can provide very accurate results for a wide variety of problems, though is computationally rather inefficient.
6 APPLICATION OF FOKKER-PLANCK-KOLMOGOROV EQUATION METHOD

The approximate solution techniques are needed for non-linear system because the exact solution method involves certain restriction which in practical systems, not usually met. In previous chapter we have discussed several methods proposed by several authors.[Cauchy, 1959],[Crandall, 1963],[Lin, 1967]. In the present chapter, we discuss the method proposed by Guo-Kang Er [2000], called Exponential closure method, which is further simplified by Rong et al.[2003].

In exponential closure method, the probability density function (PDF) of the responses of non-linear stochastic system is assumed to be an exponential function of polynomial in state variables. Special measure is taken to satisfy Fokker-Plank-Kolomogorov (FPK) equation in the weak sense of integration with the assumed PDF. The problem of evaluating the parameters in the approximate PDF finally results in solving simultaneous quadratic algebraic equations. Gaussian closure method is special case of the method. In numerical practice, method is generally suitable for the system being of polynomial type of drift and diffusion coefficient.

6.1 Problem Formulation

The problem of randomly excited non-linear multidimensional systems can be proposed in Stratonovich’s form as

\[
\frac{d}{dt} X_i = f_i(X) + g_{ij}(X)\xi_j(t), \quad i = 1,2,\ldots,n_x
\]

\[
j = 1,2,\ldots,m,
\]

Where \( X_i \) the components of system response are vectors \( X \), and \( \xi_j(t) \) are random excitations. Moreover each \( X_i \) assumed to be distributed on the entire range \((−\infty, +\infty)\), unless stated otherwise.The drift coefficient \( f_i \) and the diffusion coefficient \( g_{ij} \) are assumed to be off
polynomial type of non-linearity if \( n_x > 2 \) and their functional forms are assumed to be deterministic.

When the excitations \( W_j(t) \) are Gaussian white noises with Zero mean and cross-correlation
\[
E[W_j(t)W_k(t + \tau)] = 2\pi K_{jk}\delta(\tau),
\]
where \( \delta(\tau) \) is Dirac function and \( K_{jk} \) are constants, representing the cross spectral density of \( W_j \) and \( W_k \), the system response \( X \) is a Markov vector governed by the following FPK equation[Soong., T.T., 1973]
\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}(A_ip) - \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j}(B_{ij}p) = 0, \quad (6.2)
\]
\[
p(x, t_0|x_0, t_0) = \prod_{j=1}^{n_x} \delta(x_j - x_{j,0}), \quad (6.3)
\]
where \( p(x, t_0|x_0, t_0) \), \( x_j \) are state variables, and the first and second derivate moments \( A_i \) and \( B_{ij} \) can be derived from equation (6.1) as follows
\[
A_i(x) = f_i(x) + \pi K_{is}g_{rs}(x) \frac{\partial}{\partial x_r}g_{il}(x), \quad (6.4)
\]
\[
B_{ij}(x) = 2\pi K_{is}g_{il}(x)g_{js}(x), \quad (6.5)
\]
where \( x = \{x_1, x_2, ..., x_{n_x}\}^T \) is the state vector and Wong and Zakai’s correction term on physical system considered in equation(6.4).

In stationary state, \( p \) is independent of time and the FPK equation reduced to
\[
\frac{\partial}{\partial x_i}(A_ip) - \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j}(B_{ij}p) = 0, \quad (6.6)
\]
The stationary state solutions of FPK equation are useful in reliability and other statics analysis, so several work [Paola et al. 2002], [Er, 1999], [Cai and Lin, 1996] has been done for this approximate solution of this equation.

### 6.2 Exponential Closure Method

The method was proposed by Guo-Kang Er [2000]. The approximate solution \( \tilde{p}(x; \alpha) \) of the reduced FPK equation
\[
\tilde{p}(x; \alpha) = c \exp^{Q_n(x; \alpha)}, \quad x, \alpha \in D_x \times \mathbb{R}^{N_p} \quad (6.7)
\]
\[
\tilde{p}(x; \alpha) = 0 \quad \text{otherwise}
\]
where \( \alpha \) unknown parameter vector, \( \alpha = \{a_1, a_2, ..., a_{N_p}\} \in \mathbb{R} \) which is a \( N_p \) dimensional real space, \( N_p \) is total number of unknown parameters, \( D_x \) is a domain around the mean values of \( x, D_x \in \mathbb{R} \) and
\[
Q_n(x) = \sum_{i=1}^{n_x} a_i x_i + a_{n_x + 1} x_1^2 + a_{n_x + 2} x_1 x_2 + ... + a_{n_x(n_x+3)/2} x_1^2 + ... + a_{N_p} x_{n_x}^n \quad (6.8)
\]
which is a n-degree polynomial in $x_1, x_2, \ldots, x_n$.

Writing equation (6) in expanded form:

$$\frac{\partial A_j}{\partial x_j} p + A_j \frac{\partial p}{\partial x_j} - \frac{1}{2} \left( \frac{\partial^2 B_{ij}}{\partial x_i \partial x_j} p + \frac{\partial B_{ij}}{\partial x_j} \frac{\partial p}{\partial x_i} + \frac{\partial B_{ij}}{\partial x_i} \frac{\partial p}{\partial x_j} + B_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} \right) = 0 \quad (6.9)$$

Generally, FPK equation cannot be satisfied exactly $\tilde{p}(x; a)$ because $\tilde{p}(x; a)$ is only an approximation of $p(x)$ and the number of unknowns always limited in practice.

Now substituting $\tilde{p}(x; a)$ for $p(x)$ in equation (6.8) leads to following residual error

$$\frac{\partial A_j}{\partial x_j} \tilde{p} + A_j \frac{\partial \tilde{p}}{\partial x_j} - \frac{1}{2} \left( \frac{\partial^2 B_{ij}}{\partial x_i \partial x_j} \tilde{p} + \frac{\partial B_{ij}}{\partial x_j} \frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial B_{ij}}{\partial x_i} \frac{\partial \tilde{p}}{\partial x_j} + B_{ij} \frac{\partial^2 \tilde{p}}{\partial x_i \partial x_j} \right) = 0 \quad (6.10)$$

Substituting equation (6.6) into equation (6.9)

$$\Delta(x; a) = r(x; a) \tilde{p}(x; a)$$

where

$$\frac{\partial A_j}{\partial x_j} Q_n + A_j \frac{\partial Q_n}{\partial x_j} - \frac{1}{2} \left( \frac{\partial^2 B_{ij}}{\partial x_i \partial x_j} Q_n + \frac{\partial B_{ij}}{\partial x_j} \frac{\partial Q_n}{\partial x_i} + \frac{\partial B_{ij}}{\partial x_i} \frac{\partial Q_n}{\partial x_j} + B_{ij} \frac{\partial^2 Q_n}{\partial x_i \partial x_j} \right) = 0 \quad (6.11)$$

Because $\tilde{p}(x; a) \neq 0$ generally, therefore the only possibility for $\tilde{p}(x; a)$ to satisfy equation (6.10) is $r(x; a) = 0$. However usually $r(x; a) \neq 0$ because $\tilde{p}(x; a)$ is only an approximation of $p(x)$. In this case another case of mutually independent functions $h_k(x)$ which span space $\mathbb{R}^{N_p}$ can be introduced to make the projection of $r(x; a)$ on $R^{N_p}$ vanish, which leads to

$$\int r(x; a) h_k(x) dx = 0 \quad , k = 1, 2, \ldots, N_p \quad (6.12)$$

This means that $\tilde{p}(x; a)$ satisfies the reduced FPK equation in the weak sense of integration if $r(x; a) h_k(x)$ is integrable in $\mathbb{R}^{n_x}$.

By selecting $h_k(x)$ as $x_1^{k_1}, x_2^{k_2}, \ldots, x_n^{k_{n_x}} f_N(x)$, being $k_1, k_2, \ldots, k_{n_x} = 0, 1, 2, \ldots, N_p$. So, Equation (6.12) results in $N_p$ simultaneous equations as follows:

$$F_k(a_1, a_2, \ldots, a_{N_p}) = 0 \quad , k = 1, 2, \ldots, N_p \quad (6.13)$$

Equation (6.13) is $N_p$ quadratic non-linear algebraic equations in terms of $N_p$ undetermined parameters. The algebraic equation can be solved with any available method to determine unknown parameters.

### 6.3 Approximation Closure Method

In the present chapter, a new approximation closure method is discussed and is applied to several non-linear systems of different order of non-linearity. Approximation closure method [Rong et al., 2003] is extension and simplified version of exponential closure method. In
present approximation method, instead of solving FPK equation directly as in exponential
closure method, the approximate PDF is substituted in FPK equation and a function in terms
of x and unknown parameter \( \alpha \) is factored out. After that, making the coefficient of residual
error vanishing, the unknown parameter \( \alpha \) can be determined.

### 6.4 Numerical Results for First Order Non-linear System

First order non-linear can be a scalar diffusion process. Considering a stochastic differential
equation governing a scalar diffusion process, can be written as

$$\dot{X} - \frac{1}{2}(X - X^3 - \varepsilon X^5) = W(t) \tag{6.14}$$

where \( W(t) \) is a Gaussian white noise with zero mean and correlation function
\( E[W(t)W(t+\tau)] = 2\pi S_0 \delta(\tau) \) with \( S_0 \) being the spectral density of \( W(t) \), \( \varepsilon \) is constant. For
\( S_0 = \frac{1}{\pi} \), we know that the stationary probability density function of Markov process \( X \)
is governed by following stationary FPK equation:

$$\frac{d}{dx} \left[ \frac{1}{2}(x - x^3 - \varepsilon x^5)p(x) \right] - \frac{d^2p(x)}{dx^2} = 0 \tag{6.15}$$

Exact stationary solution of equation (6.14) can be given as

$$p(x) = C \exp\left(\frac{x^2}{4} - \frac{x^4}{8} - \frac{\varepsilon x^6}{12}\right) \tag{6.16}$$

where \( C \) is normalization constant.

Now assuming the approximate solution has form [Er, 2000]

$$p_n(x) = C \exp(\sum_{i=1}^{n} a_i x^i) \tag{6.17}$$

where \( C \) is normalization constant, \( a_i \) and \( n \) are unknown parameters. generally the equation
(6.15) cannot be satisfied exactly by \( p_n(x) \) because \( p_n(x) \) is only an approximation of \( p(x) \).So for solving FPK equation, let us substitute \( p_n(x) \) in equation (6.15) in place of \( p(x) \)
which leads to following residual error.

$$\frac{d}{dx} \left[ \frac{1}{2}(x - x^3 - \varepsilon x^5)p_n(x) \right] - \frac{d^2p_n(x)}{dx^2} = p_n(x)h_n(x) \tag{6.18}$$

where \( h_n(x) \) is a polynomial function of the variable \( x \). Since \( p_n(x) \neq 0 \), so the only
possibility to satisfy equation (6.15) is \( h_n(x) = 0 \). however, usually \( h_n(x) \neq 0 \), because
\( p_n(x) \) is only an approximation of \( p(x) \). This case Coefficients of \( x^i \) in \( h_n(x) \) must vanish.

That is:

$$F_i(a_1, a_2, ..., a_n) = 0, \quad i = 0,1,2,...,n-1 \tag{6.19}$$
This means $p_n(x)$ satisfies reduced FPK equation (6.14) in the weak sense that the coefficients of low order power of $x^i$ in $h_n(x)$ vanish.

**Case 1:** $n = 2$

When, $n = 2$ the approximate probability density function can be written as:

$$ p_n(x) = C \exp(a_1 x + a_2 x^2) \quad (6.20) $$

Substituting $p_2(x)$ for $p(x)$ in equation (6.15) leads to following error:

$$ \frac{d}{dx} \left[ \frac{1}{2} (x - x^3 - \varepsilon x^5) p_2(x) \right] - \frac{d^2 p_2(x)}{dx^2} = p_2(x) h_2(x) \quad (6.21) $$

$$ h_2(x) = (0.5 - a_1^2 - 2a_1 a_2) + \cdots \quad (6.22) $$

Making the coefficients $F_i(a_1, a_2, \ldots, a_4)$ of $x^i$ leads to

$$ a_1 = 0 \text{ and } a_2 = -0.5 \quad (6.23) $$

**Case 2:** $n = 4$

The approximate probability density function for $n = 4$

$$ p_4(x) = C \exp(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \quad (6.24) $$

Substituting $p_4(x)$ for $p(x)$ in equation (6.15) leads to following error:

$$ \frac{d}{dx} \left[ \frac{1}{2} (x - x^3 - \varepsilon x^5) p_4(x) \right] - \frac{d^2 p_4(x)}{dx^2} = p_4(x) h_4(x) \quad (6.25) $$

Solving the equation (6.25) and collect the coefficient:

$$ h_4(x) = (0.5 - a_1^2 - 2a_2) + (0.5a_1 - 2a_1 a_2 - 6a_3)x + $$

$$ (-1.5 + a_2 - 4a_2^2 - 6a_1 a_3 - 12a_4)x^2 + (1.5a_3 - 12a_2 a_2 - 0.5a_1 - 8a_1 a_4)x^3 + \cdots \quad (6.26) $$

making the coefficients $F_i(a_1, a_2, \ldots, a_4)$ of $x^i, i = 0, 1, 2, 3$ in $h_4(x)$ vanishing leads to

$$ \begin{cases} 
(0.5 - a_1^2 - 2a_2) = 0 \\
(0.5a_1 - 2a_1 a_2 - 6a_3) = 0 \\
(-1.5 + a_2 - 4a_2^2 - 6a_1 a_3 - 12a_4) = 0 \\
(1.5a_3 - 12a_2 a_2 - 0.5a_1 - 8a_1 a_4) = 0
\end{cases} \quad (6.27) $$

The solution of 6 algebraic equations is as follows

$$ a_1 = 0, \quad a_2 = \frac{1}{4}, \quad a_3 = 0, \quad a_4 = -\frac{1}{8} \quad (6.28) $$

This follows the approximate probability density function can written as:

$$ p_4(x) = C \exp\left(\frac{1}{4} x^2 - \frac{1}{4} x^4\right) \quad (6.29) $$

**Case 3:** $n = 6$

The approximate probability density function for $n = 6$
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\[ p_6(x) = C \exp(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6) \]  

(6.30)

Substituting \( p_6(x) \) for \( p(x) \) in equation (6.15) leads to following error:

\[ \frac{d}{dx} \left[ \frac{1}{2} (x - x^3 - \epsilon x^5) p_6(x) \right] - \frac{d^2 p_6(x)}{dx^2} = p_6(x) h_6(x) \]  

(6.31)

Solving equation (above) for \( h_6(x) \) gives following coefficients

\[ a_1 = 0, \quad a_2 = \frac{1}{4}, \quad a_3 = 0, \quad a_4 = -\frac{1}{8}, \quad a_5 = 0, \quad a_6 = \frac{1}{240} \]  

(6.32)

This gives the approximate stationary probability density function of system given by equation (6.14)

\[ p_6(x) = C \exp \left( \frac{1}{4} x^2 - \frac{1}{4} x^4 + \frac{1}{240} x^6 \right) \]  

(6.33)

6.5 Numerical Result for Second Order Non-linear System

**Oscillator 1.** Considering the following oscillator with additive Gaussian white noise excitations

\[ \ddot{X} + \alpha \dot{X} + X + \epsilon_1 X^3 + \epsilon_2 X^5 = \sqrt{\alpha} W(t) \]  

(6.34)

where \( \epsilon_1 \) and \( \epsilon_2 \) are parameters denoting the degree of non-linearity and \( W(t) \) is Gaussian white noise with correlation function \( E[W(t)W(t + \tau)] = 2\delta(\tau) \). The equation (6.34) is normalized so that when \( \epsilon_1 = \epsilon_2 = 0 \), the stationary mean-square displacement and velocity both equal to 1. The stationary probability density function \( p(x_1, x_2) \) can be given as:

\[ p(x_1, x_2) = C \exp \left( -\frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 - \frac{\epsilon_1}{4} x_1^4 - \frac{\epsilon_2}{6} x_2^6 \right), \]  

(6.35)

where \( C \) is normalization constant.

Markov vector \( X = X_1 \) and \( \dot{X} = X_2 \) are governed by the following FPK equation:

\[ -x_2 \frac{\partial p(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ (\alpha x_2 + x_1 + \epsilon_1 x_1^3 + \epsilon_2 x_1^5) p(x_1, x_2) \right] + \frac{\partial^2 p(x_1, x_2)}{\partial x_2^2} = 0 \]  

(6.36)

Now the approximate closure method [Rong. et al. 2003] is used to solve this equation. For this \( p(x_1, x_2) \) can be assumed as:

\[ p_n(x_1, x_2) = C \exp(a_1 x_1 + a_2 x_2 + a_3 x_1^2 + \cdots + a_{n_p} x_1^n + a_{n_p+1} x_1^{n-1} x_2 + a_{n_p+n} x_2^n) \]  

(6.37)

where \( n_p = n(n + 1)/2 \)

**Case 1:** \( n = 2 \),

The approximate stationary probability density function for \( n = 2 \) can be assumed as:
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\[ p_2(x_1, x_2) = C \exp \left( a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 \right) \]  

(6.38)

Substituting equation (6.37) in FPK equation (6.35) which satisfy,

\[ -x_2 \frac{\partial p_2(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ \left( \alpha x_2 + x_1 + \varepsilon_1 x_1^3 + \varepsilon_2 x_1^5 \right) p_2(x_1, x_2) \right] + \frac{\partial^2 p_2(x_1, x_2)}{\partial x_2^2} = p_2(x_1, x_2) h_2(x_1, x_2) \]  

(6.39)

for \( \varepsilon_1 = \varepsilon_2 = 5 \), making coefficient of \( F_i \left( a_1, a_2, \ldots, a_{n+n_p} \right) = 0 \), \( i = 1, 2, \ldots, n + n_p \) results:

\[ a_1 = 0, a_2 = 0, a_3 = -\frac{1}{2}, a_4 = 0, a_5 = -\frac{1}{2} \]  

(6.40)

**Case 2: \( n = 4 \),**

The approximate stationary probability density function for \( n = 4 \) can be assumed as:

\[ p_4(x_1, x_2) = C \exp \left( a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 + a_6 x_1^3 + a_7 x_1^5 x_2 + a_8 x_1 x_2^2 + a_9 x_1 x_2^2 + a_{10} x_1^3 + a_{11} x_1^5 + a_{12} x_1 x_2 + a_{13} x_1 x_2^3 + a_{14} x_2^4 \right) \]  

(6.41)

Substituting Equation (6.40) in FPK equation (6.36) and solving for \( h_4(x_1, x_2) \)

\[ -x_2 \frac{\partial p_4(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ \left( \alpha x_2 + x_1 + \varepsilon_1 x_1^3 + \varepsilon_2 x_1^5 \right) p_4(x_1, x_2) \right] + \frac{\partial^2 p_4(x_1, x_2)}{\partial x_2^2} = p_4(x_1, x_2) h_4(x_1, x_2) \]  

(6.42)

For \( \varepsilon_1 = \varepsilon_2 = 5 \), making coefficient of \( F_i \left( a_1, a_2, \ldots, a_{n+n_p} \right) = 0 \), \( i = 1, 2, \ldots, n + n_p \) results:

\[ a_1 = a_2 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{12} = 0, \]
\[ a_3 = -\frac{1}{2}, a_5 = -\frac{1}{2}, a_{11} = -\frac{5}{4} \]  

(6.43)

Substituting coefficient in equation (6.40) gives stationary approximate probability density function of second order non-linear system given by equation (6.33):

\[ p_4(x_1, x_2) = C \exp \left( -\frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 - \frac{\varepsilon_1}{4} x_1^4 \right) \]  

(6.44)

**Case 3: \( n = 6 \),**

For \( n = 6 \), leads to \( n_p = n(n + 1)/2 = 21 \), approximate probability density function is:

\[ p_6(x_1, x_2) = C \exp\left( a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 + \cdots + a_{21} x_1^6 + \cdots + a_{27} x_2^6 \right) \]  

(6.45)

Equation (6.45) contains 27 unknown variables, substituting equation (6.43) in equation (6.35) leads to:
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\[ -x_2 \frac{\partial p_6(x_1,x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ (\alpha x_2 + x_1 + \varepsilon_1 x_1^3 + \varepsilon_2 x_1^5) p_6(x_1,x_2) \right] + \frac{\partial^2 p_6(x_1,x_2)}{\partial x_2^2} = \]

\[ p_6(x_1,x_2) h_6(x_1,x_2) \] \hspace{1cm} (6.46)

Solving equation (6.46) for \( h_6(x_1,x_2) \) and for \( \varepsilon_1 = \varepsilon_2 = 5 \), making coefficient of \( F_i(a_1,a_2,\ldots,a_{21}) = 0, \quad i = 1,2,\ldots,21 \) results:

\[
\begin{align*}
 a_1 &= a_2 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} \\
 &= a_{19} = a_{20} = a_{22} = a_{23} = a_{24} = a_{25} = a_{26} = a_{27} = 0, \\
 a_3 &= -0.5, \quad a_5 = -0.5, \quad a_{11} = -1.25, \quad a_{21} = -0.833 \hspace{1cm} (6.47)
\end{align*}
\]

Equation (6.47) gives the stationary approximate probability density function as:

\[ p_6(x_1,x_2) = C \exp\left(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{\varepsilon_1}{4} x_1^4 + \frac{\varepsilon_2}{6} x_1^6\right) \] \hspace{1cm} (6.48)

**Case 4: \( n = 8 \),**

For \( n = 8 \), leads to \( n_p = n(n+1)/2 = 36 \), approximate probability density function is:

\[ p_8(x_1,x_2) = C \exp(a_1x_1 + a_2x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2 + a_{21}x_1^6 + \cdots + a_{41}x_1^8 + \cdots + a_{44}x_2^8) \] \hspace{1cm} (6.49)

Equation (6.49) contains 44 unknown variables, substituting equation (6.49) in equation (6.36) lead to:

\[ -x_2 \frac{\partial p_8(x_1,x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ (\alpha x_2 + x_1 + \varepsilon_1 x_1^3 + \varepsilon_2 x_1^5) p_8(x_1,x_2) \right] + \frac{\partial^2 p_8(x_1,x_2)}{\partial x_2^2} = \]

\[ p_8(x_1,x_2) h_8(x_1,x_2) \] \hspace{1cm} (6.50)

Solving equation (6.50) for \( h_8(x_1,x_2) \) and for \( \varepsilon_1 = \varepsilon_2 = 5 \), making coefficient of \( F_i(a_1,a_2,\ldots,a_{44}) = 0, \quad i = 1,2,\ldots,44 \) results:

\[
\begin{align*}
 a_1 &= a_2 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{10} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17} = a_{18} \\
 &= a_{19} = a_{20} = a_{22} = a_{23} = a_{24} = a_{25} = a_{26} = a_{27} = \cdots = a_{44} = 0, \\
 a_3 &= -0.5, \quad a_5 = -0.5, \quad a_{11} = -1.25, \quad a_{21} = -0.833, a_{41} = 0 \hspace{1cm} (6.51)
\end{align*}
\]

Equation (6.51) gives the stationary approximate probability density function as:

\[ p_8(x_1,x_2) = C \exp\left(-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{\varepsilon_1}{4} x_1^4 + \frac{\varepsilon_2}{6} x_1^6\right) \] \hspace{1cm} (6.52)

**Oscillator 2.** Considering Duffing oscillator investigated previously by Crandall [1980]:

\[ \ddot{X} + \eta \dot{X} + (X + \varepsilon X^3) = \sqrt{\eta} W(t) \] \hspace{1cm} (6.53)
where $\varepsilon$ is a parameter representing the degree of non-linearity and $W(t)$ is a Gaussian white noise with correlation function

$$E[W(t)W(t+\tau)] = 2\delta(\tau) \quad (6.54)$$

Equation (6.50) and equation (6.52) have been normalized so that when $\varepsilon = 0$ the stationary mean square displacement and velocity is both equal to 1.

The stationary probability density of the displacement $X = X_1$ and velocity $\dot{X} = X_2$ is known to be:

$$p(x_1, x_2) = C \exp\left(-\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 - \frac{\varepsilon}{4}x_1^4\right) \quad (6.55)$$

where $C$ is a normalization constant given by

$$C = \frac{\varepsilon}{\pi} \sqrt{\exp\left(\frac{1}{8\varepsilon}\right)K_{1/4}\left(\frac{1}{8\varepsilon}\right)}^{-1} \quad (6.56)$$

in which $K_{1/4}$ is a modified Bessel function [Gradshteyn and Ryzhik, 1980].

Following the approximation closure method [Rong et al. 2003], the Fokker-Planck-Kolomogorov can be written as:

$$-x_2 \frac{\partial p_n(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left(\eta x_2 + x_1 + \varepsilon x_1^3\right)p_n(x_1, x_2)\right] + \frac{\partial^2 p_n(x_1, x_2)}{\partial x_2^2} = 0 \quad (6.57)$$

In equation (6.57) for different values of $n$, the PDF approximation leads to following expression:

**Case 1.** $n = 2,$

$$-x_2 \frac{\partial p_2(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left(\eta x_2 + x_1 + \varepsilon x_1^3\right)p_2(x_1, x_2)\right] + \frac{\partial^2 p_2(x_1, x_2)}{\partial x_2^2} = p_2(x_1, x_2)h_2(x_1, x_2) \quad (6.58)$$

where $p_2(x_1, x_2) = C \exp(a_1x_1 + a_2x_2 + a_3x_1^2 + a_4x_1x_2 + a_5x_2^2)$

Solving equation (6.58) for $h_2(x_1, x_2)$ and making coefficients $x^i$ for $i = 1, 2, \ldots 5$, leads to:

$$a_1 = 0, a_2 = 0, a_3 = -\frac{1}{2}, a_4 = 0, a_5 = -\frac{1}{2} \quad (6.59)$$

So Approximate probability density function can be given as:

$$p_2(x_1, x_2) = C \exp\left(-\frac{1}{2}x_2^2 - \frac{1}{2}x_1^2\right) \quad (6.60)$$

where $C$ is a normalization constant.

**Case 2.** $n = 4,$

The FPK equation in terms of approximate probability density function

$$-x_2 \frac{\partial p_4(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left(\eta x_2 + x_1 + \varepsilon x_1^3\right)p_4(x_1, x_2)\right] + \frac{\partial^2 p_4(x_1, x_2)}{\partial x_2^2} = p_4(x_1, x_2)h_4(x_1, x_2) \quad (6.61)$$
where \( p_4(x_1, x_2) \) can be written as:

\[
p_4(x_1, x_2) = C \exp \left( a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_1 x_2 + a_5 x_2^2 + a_6 x_1^3 + a_7 x_1^2 x_2 + a_8 x_2^3 + a_9 x_1^2 x_2 + a_{10} x_1^3 + a_{11} x_1^4 + a_{12} x_1^3 x_2 + a_{13} x_1 x_2^3 + a_{14} x_2^4 \right)
\]

(6.62)

Solving equation (6.61) for \( h_2(x_1, x_2) \) and for \( \varepsilon = 1 \), making coefficients \( x^i \) for \( i = 1,2, \ldots 14 \), leads to:

\[
a_1 = a_2 = a_4 = a_6 = a_7 = a_9 = a_{10} = a_{12} = 0,
\]

\[
a_3 = -0.5, a_5 = 0.5, a_{11} = 0.25
\]

(6.63)

So Approximate probability density function can be given as:

\[
p_4(x_1, x_2) = C \exp \left( -\frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 - \frac{\varepsilon}{4} x_1^4 \right)
\]

(6.64)

**Oscillator 3.** Considering another second non-linear oscillator under additive Gaussian white noise given by following by equation:

\[
\ddot{X} + \beta \dot{X} + X + X^3 + X^5 = W(t)
\]

(6.65)

where \( W(t) \) is a Gaussian white noise with zero mean with correlation function \( E[W(t)W(t + \tau)] = 2\delta(\tau) \), the stationary probability density function \( p(x_1, x_2) \) of Markov vector \( X = X_1 \) and \( \dot{X} = X_2 \) is governed by the following FPK equation

\[
-x_2 \frac{\partial p(x_1, x_2)}{\partial x_1} + \frac{\partial}{\partial x_2} \left[ (\beta x_2 + x_1 + x_1^3 + x_1^5) p(x_1, x_2) \right] + \frac{\partial^2 p(x_1, x_2)}{\partial x_2^2} = 0
\]

(6.66)

The exact stationary probability density function obtained as [Dimentberg, 1982]

\[
p(x_1, x_2) = C \exp \left[ -\beta \left( \frac{1}{2} x_2^2 + \frac{1}{4} x_1^3 + \frac{1}{6} x_1^5 \right) \right],
\]

(6.67)

Now using the approximation closure method to solve equation (6.66), the approximate solution can be of form:

\[
p_n(x_1, x_2) = C \exp (a_1 x_1 + a_2 x_2 + a_3 x_1^2 + \ldots + a_{n_p} x_1^n + a_{n_p+1} x_1^{n-1} x_2 + a_{n_p+n} x_2^n)
\]

(6.68)

where \( n_p = n(n + 1)/2 \)

Here solution \( p_n(x_1, x_2) \) can be assumed to satisfy FPK equation weak sense of integration as follows.

For \( \beta = 1 \), solving equation (6.69) for \( h_n(x_1, x_2) \) leading to coefficient of \( x^i \) in \( h_n(x_1, x_2) \) result into following coefficients for different values of \( n \):

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Approximate Stationary probability density function can be given as:

\[ n = 2, \quad p_2(x_1, x_2) = C \exp \left[ -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 \right], \quad (6.69) \]

\[ n = 4, \quad p_4(x_1, x_2) = C \exp \left[ -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4 \right], \quad (6.70) \]

\[ n = 6, \quad p_6(x_1, x_2) = C \exp \left[ -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4 - \frac{1}{6} x_1^6 \right], \quad (6.71) \]

\[ n = 8, \quad p_8(x_1, x_2) = C \exp \left[ -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4 - \frac{1}{6} x_1^6 \right], \quad (6.72) \]

\[ n = 10, \quad p_{10}(x_1, x_2) = C \exp \left[ -\frac{1}{2} x_2^2 - \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4 - \frac{1}{6} x_1^6 \right], \quad (6.73) \]

6.6 Results and discussions

Figure (6.1) is approximate probability density function plot of scalar diffusion process represented by equation (6.13). This approximate probability density function is obtained using equation (6.14)-(6.16). In Equation (6.13), \( \varepsilon \) represent the degree of nonlinearity in the process. Following approximate closure method, firstly four degree polynomial is assumed as approximate PDF and latter six degree polynomial.

Numerical results, obtained from equations (6.14)-(6.32), showed that the exact solution is obtained for any value of \( \varepsilon \) in the case when \( n = 6 \). In the case when \( n = 4 \), exact solution is obtained from equation (6.23)-(6.27) if \( \varepsilon = 0 \). If \( \varepsilon = 0.05 \), the exact and approximate PDFs of \( X \) is shown in figure (6.1). From figure (6.1), it can be observed that values of PDFs in the tails are very close to exact solution. The values of PDFs at the tail of approximate function are extremely important in reliability analysis.
Figure (6.2) represent the approximate PDF of $X_1$ for $n = 2, 4$ and $6$ and their comparison with exact PDF for $\varepsilon_1 = \varepsilon_2 = 5$. Here in equation (6.33) $\varepsilon_1$ and $\varepsilon_2$ represent degree of non-linearity. So, higher values of $\varepsilon_1$ or $\varepsilon_2$ represent highly non-linear oscillator. In Figure (6.2), it can be clearly observed that the approximate PDF obtained from $n = 4$ are much better that the results obtained from $n = 2$, which represent Gaussian closure. Numerical results furthermore obtained from $n = 6$, gives the exact probability density function. For $n = 6$, exact PDF is obtained for any values of $\varepsilon_1$ and $\varepsilon_2$.

Furthermore, numerical results for higher values of $n > 6$, result the exact PDF as obtained by $n = 6$, regardless of values of $\varepsilon_1$ and $\varepsilon_2$.

Figure (6.3a),(6.3b),(6.3c),(6.3d), represent three dimensional approximate probability density function representation for different values of $n$. It can be observed in Figure 3, that the tail becoming broader, as the values of $n$, increases. Furthermore figure (6.3c) and (6.3d) are exactly same exact PDF is obtained by value of $n = 6$, so further approximation for PDF gives same result regardless of degree of non-linearity.
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Figure (6.3a). The PDF of oscillator 1 for \( n = 2 \)  

Figure (6.3b). The PDF of oscillator 1 for \( n = 4 \)  

Figure (6.3c). The PDF of oscillator 1 for \( n = 6 \)  

Figure (6.3d). The PDF of oscillator 1 for \( n = 8 \)  

Figure (6.4). The PDF of oscillator 1 for \( n = 2, 4 \)  

Figure (6.44) represent the approximate probability density function of Duffing oscillator given by equation (6.51) previously investigated by Crandall [1980]. Here result obtained by approximate closure method is compared with obtained by Crandall. Numerical results obtained for different values of \( n \) gives the same approximate PDFs and statistical moments as obtained by Crandall. Figure (6.4) gives the approximate PDFs of \( X_1 \) for \( n = 2, 4 \). For \( \varepsilon = 1, n = 2 \), which is Gaussian closure gives the same results as obtained in section (5.5)
For $\varepsilon = 1$ Exact PDF is obtained by $n = 4$ approximation. Furthermore for higher values of $n$ yield same approximate PDF as obtained by $n = 4$.

After analyzing all previous numerical results for different order non-linear oscillator, one very important point can be concluded that approximation for $n$, value depends upon the highest power of displacement state variable in system equation. Numerical results show when $n$ is equal to highest power of displacement state variable in system equation yield approximate PDF equal to the exact obtained by [Dimentberg, 1982].

Figure (6.4). The PDFs of oscillator 2, Gaussian closure $n = 2$; exact $n = 4$.

Figure (6.5) results the PDF of $X_1$ considered by second order non-linear oscillator 3 under additive Gaussian white noise given by equation (6.63). The $\beta$ represent the non-linearity in the systems. PDF for $n = 2$ obtained by solving equation (6.64), for $n = 4$ and $n = 6$ by solving equation (6.66) and equation (6.67) respectively. Numerical results are also obtained for higher values of $n$ ($n > 6$) by solving equations (6.67). Numerical results obtained for this system is shown in figure(6.5). Here again, we can observed much better results are obtained by $n = 4$ in comparison to Gaussian closure ($n = 2$). From figure (6.5), it can be easily observed that approximate PDF is coincide with exact one for $n = 6$. Moreover, for $n = 6$, the numerical results obtained are same for any value of $\beta$. So this procedure is also suitable for strongly damped systems.
Figure (6.5). The PDF of oscillator 3, \( n = 2; n = 4; \) \textit{exact} \( n = 6 \)

Figure (6.6a), (6.6b), (6.6c), (6.6d) are the three dimensional PDF representation for different values of \( n = 2, 4, 6 \) and 8. Figure (6.6c) and figure (6.6d) are PDFs for \( n = 6 \) and \( n = 8 \) respectively, which exactly same. Numerical results are also obtained for higher value of \( n (n > 8) \) represented by equation (6.70). But all PDFs obtained by higher value of \( n (n > 6) \) yield same PDF as obtained by \( n = 6 \), which coincide exact PDF irrespective of order of non-linearity in the system.

\[
p_2(x_1, x_2)
\]

Figure (6.6a). The PDF of oscillator 3 for \( n = 2 \)
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Figure (6.6b). The PDF of oscillator 3 for $n = 4$

$P_6(x_1, x_2)$

Figure (6.6c). The PDF of oscillator 3 for $n = 6$

$P_6(x_1, x_2)$

Figure (6.6d). The PDF of oscillator 3 for $n = 8$
Figure (6.6b) which is much closer to the exact PDF in comparison to figures (6.6a) which represent the Gaussian closure.

In the present method FPK equation is solved with assumed exponential function of polynomial of several degrees in such a way that assumed PDF satisfy FPK equation weak sense that the coefficient of low order power of state variables in residual error must vanish. This method is not only valid for single degree of freedom polynomial random systems but also for multiple degree of freedom polynomial random systems. Though the choice of \( n \) depends upon the numerical calculation experience but from above numerical results, one can easily choose the values of \( n \) which coincide with exact PDF of system.
7 CONCLUSIONS

In thesis, stochastically excited nonlinear dynamic systems were investigated. Within the scope of this investigation theme, the work can be characterized by two main line of research. The first line of research focuses on the study, development and qualification of response approximation methods. Herein, characteristics of these methods, such as efficiency, accuracy and applicability, jointly determine the quality of the method. Once such response approximation methods are available, attention can be given to the second line of research. Herein, the focus lies on gaining thorough understanding of specifically nonlinear, stochastic response phenomenon.

The main conclusions are summarized below:

- In Many cases of complicated nonlinearities, such as non as non-differentiable functions or hysteresis or multi-degree-of-systems with polynomial non-linearities, when the standard closure techniques (cumulant or exponential or approximate closure method) fails or lead to very complicated formulas, from a numerical point of view, then statistical or equivalent linearization are the only suitable analytical techniques for response analysis.(Chapter 6)

- The accuracy of square response characteristics obtained by statistical linearization in reviewed applications for proposed sets of parameters was satisfactory. (The error of approximation was smaller than 25 %.)

- Although the application of statistical and equivalent linearization to response analysis of nonlinear multi-degree-of-freedom systems in comparison to other closure techniques leads to a smaller number of moment equations; nevertheless, it may be still an extremely large number and it bear difficulties with numerical calculations. These problems are very important if a considered system has thousand of degree of freedom.

- In the present thesis, a wide variety of successful methods were discussed, such as closure techniques, stochastic averaging, method of weighted residuals, however, an important
shortcoming of these methods is reflected by the fact that these methods only provide
information on the probability distribution of the response. In fact this information represents
only partly the information on the response statistics; frequency domain information is not
provided by the methods mentioned above. Furthermore the applicability of these methods in
terms of variety of systems and stochastic excitations that can be investigated by applications
of these methods is rather limited. (Chapter 5)

• In the chapter 6, a closure technique was adopted for obtaining approximate probability
density function of various nonlinear oscillators. These oscillators belong to various orders of
system (first and second) as well different levels of non-linearity. This closure technique is a
method that assumes that the approximate PDF is special kind of function like here it is
exponential function of polynomial in state variables. In present work, method is extended for
exponential function of higher order polynomial. From numerical result obtained in chapters
6, it can be stated that the choice of assumed order of exponential function of polynomial, for
example $n$ here depends upon the maximum of power of displacement state variable in the
system equation. All the higher order assumed polynomial like $n = 8,10$ in oscillator 3 yield
the same approximate probability as given by $n = 6$, where is highest power of displacement
variable in oscillator is 5. Similar trends also obtained in other two oscillators. In other word,
to conclude, present study presents a good choice of approximation of order of polynomial
which exactly coincides with the exact stationary probability density function of nonlinear
systems.
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